From "sense of number" to "sense of magnitude": The role of continuous magnitudes in numerical cognition

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Abstract: In this review, we are pitting two theories against each other: the more accepted theory, the *number sense theory*, suggesting that a sense of number is innate and non-symbolic numerosity is being processed independently of continuous magnitudes (e.g., size, area, and density); and the newly emerging theory suggesting that (1) both numerosities and continuous magnitudes are processed holistically when comparing numerosities and (2) a sense of number might not be innate. In the first part of this review, we discuss the number sense theory. Against this background, we demonstrate how the natural correlation between numerosities and continuous magnitudes makes it nearly impossible to study non-symbolic numerosity processing in isolation from continuous magnitudes, and therefore, the results of behavioral and imaging studies with infants, adults, and animals can be explained, at least in part, by relying on continuous magnitudes. In the second part, we explain the *sense of magnitude theory* and review studies that directly demonstrate that continuous magnitudes are more automatic and basic than numerosities. Finally, we present outstanding questions. Our conclusion is that there is not enough convincing evidence to support the number sense theory anymore. Therefore, we encourage researchers not to assume that number sense is simply innate, but to put this hypothesis to the test and consider whether such an assumption is even testable in the light of the correlation of numerosity and continuous magnitudes.

Keywords: animal studies; cognitive control; continuous magnitudes; functional studies; holistic processing; number sense; numerical cognition; numerosities

1. Introduction

We all use mathematics in everyday life, whether to calculate the change given to us in the store, to tell time, or to choose the shortest line in the grocery store. Not only humans use math; numerical cognition abilities are important for survival across species: fish join the larger shoal to reduce their chances of being eaten, bees can identify flowers by counting the number of their petals, and so forth (Agrillo et al. 2016; Gross et al. 2009; Pisa & Agrillo 2008). These efforts have led to the widely accepted view

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that humans and animals are born with a "sense for number" – the ability to perceive, manipulate, and understand numerosities (Cantlon et al. 2009b; Dehaene 1997; Feigenson et al. 2004) – and that this ability, along with general cognitive abilities, enables humans to understand more complex mathematical principles. Recently, however, it has been suggested that perceiving numerosities might not be as innate and automatic as previously thought. Instead, a new line of theories suggest that a "sense of magnitude" that enables discrimination between different continuous magnitudes, such as between the densities of two groups of apples or the total surface areas of two pizza trays, is even more basic and automatic than a sense of numbers (Henik et al. 2017; Gebuis & Reynvoet 2012a; Leibovich & Ansari 2016; Leibovich et al. 2015; 2016a; Mix et al. 2002a).

In this review, we introduce the mainstream theories in the field of numerical cognition (e.g., *number sense theory*) and the studies on which these theories are based. Specifically, we concentrate on studies that employed comparison of non-symbolic stimuli (i.e., groups of items) and point out how the results of such studies can be explained by a sense of magnitude and not necessarily number. Later, we explain the *sense of magnitude theory* and review studies that directly demonstrate that continuous magnitudes are more automatic and basic than numerosities. The idea of a sense of magnitude, or a general system that processes all magnitudes, is not new (for a review, see Mix & Sandhofer [2007]). In fact, this idea was popular before the 1990s (e.g., Meck &

Church 1983; Moyer & Landauer 1967) and was neglected with the increased popularity of the number sense theory. The current review integrates new evidence supporting this idea. Importantly, we review empirical evidence suggesting that processing of continuous magnitudes, and not number (i.e., discrete numerosity), is innate and automatic. We also discuss the role of domain-general factors such as language and cognitive control in the acquisition of the concept of number. Lastly, we discuss possible implications of the sense of magnitude theory on research in numerical cognition and the diagnosis and amelioration of learning disabilities specific for math (i.e., developmental dyscalculia [DD]).

2. Mainstream theories in the field of numerical cognition

Because numbers play an important role in our lives, research has long been occupied with the cognitive structures underlying numerical cognition. We present three major theories that are at the forefront of the field and have set the tone for research in numerical cognition and the procedures for diagnosis of math learning difficulties (MLDs) and DD. The theories are summarized in Figure 1.

In his seminal book, Dehaene (1997) suggested that humans and animals are born with a "sense of number," the ability to perceive, understand, and manipulate numerosities. For example, when encountering six strawberries, we can sense their "sixness" similarly to the way we sense their redness (Burr & Ross 2008; Nieder & Dehaene 2009), suggesting that perceiving numerosities is as basic as perceiving colors. Dehaene and Changeux (1993) suggested a computational model explaining the process of numerosity estimation. According to this model, items are first represented spatially, according to their location. These locations are then mapped onto a topographic map. This map codes only locations and ignores all other features of the items, including continuous magnitude (like the size of an individual item). Finally, specialized neurons sum the numerosities from this map, allowing us to estimate the numerosity of a heterogeneous group of items. This model was later supported by further evidence in the work of Verguts and Fias (2004) (see Fig. 1A).

Feigenson et al. (2004) expanded on the original concept of the number sense. They proposed the existence of two distinct core systems of numerical representation in humans and animals; core system 1 represents large numerosities (>4), and core system 2 represents small numerosities (1–4) (see Fig. 1B). Both core systems are considered cross modal and cross species. Core system 1 forms abstract and approximate representations of large numerosities. This system allows numerical discrimination from infancy and recognition of ordinal relationships. This system's discrimination depends on the ratio between the quantities, *regardless* of continuous magnitudes; as we get older, the sensitivity of this system increases, and we can discriminate relatively similar numerosities. For example, 6-month-old infants can discriminate numerosities of ratio 1:2 (e.g., 20 from 40 items) but not 2:3, whereas adults can discriminate 7:8 (Barth et al. 2003). This is consistent with Weber's law, which states that the difference in intensity needed to detect a difference between two stimuli (the difference between
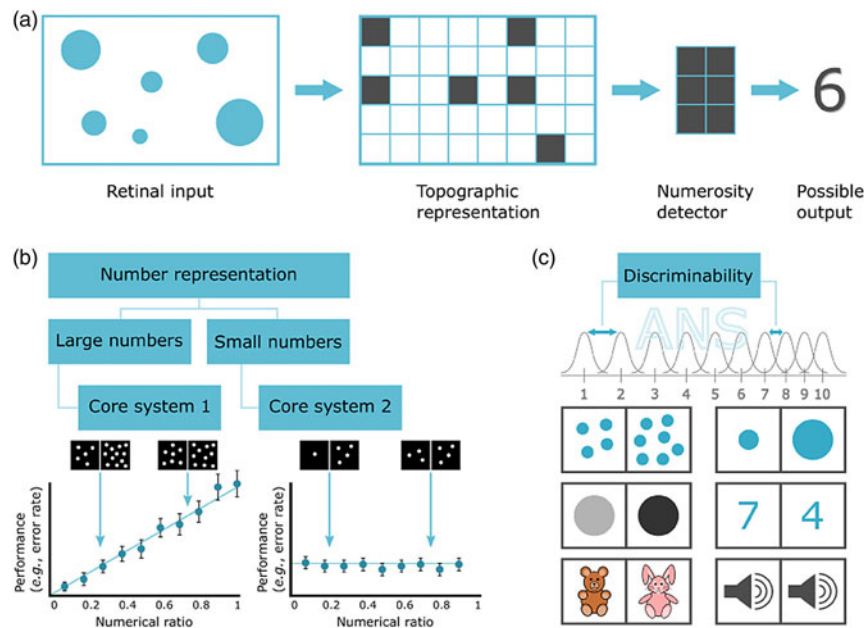
Figure 1. Main theories in numerical cognition (based on Leibovich et al. 2016a). (A) The model suggested by Dehaene and Changeux (1993), emphasizing the existence of an innate "numerosity detector" that sums numerosities regardless of continuous magnitudes. (B) The *two core systems theory* (Feigenson et al. 2004) separating processing of small and large numbers; the main difference is ratio-dependent performance in core system 1 and violation of this dependency in core system 2. (C) The approximate number system (ANS) theory suggesting that different magnitudes (examples in the figure) are being processed by an approximate analog representation of numbers.

two numerosities, for example) is proportional to the intensities of the stimuli. Studies that reported that numerical ratio modulates performance even when numbers are presented as two symbolic numerosities (Moyer & Landauer 1967) led Feigenson et al. to claim that the system allowing representation of non-symbolic numerosities is later used for complex mathematical thinking.

Core system 2 forms exact representations of small numerosities. Contrary to system 1, this system is sensitive to continuous magnitudes. In infants, this system is limited to processing and computation of three objects at most, and in adulthood, it reaches four objects. This ability to quickly, efficiently, and accurately identify small numerosities is called *subitizing* (Kaufman & Lord 1949). Performance in this range is not ratio dependent. For example, infants can successfully discriminate between two versus three items but not between four versus six, despite the identical ratio (Feigenson & Carey 2003). There is evidence for a connection between this system and mathematical abilities; Ashkenazi et al. (2013) found a deficit in subitizing abilities specific to children with DD. The existence of these two core systems might explain why some mathematical abilities are basic and intuitive, whereas others are considered difficult to acquire; estimations utilize core system 1, and exact numerical judgments in the subitizing range utilize core system 2. Neither system, however, is equipped to deal with exact calculations above the subitizing range (Feigenson et al. 2004). The studies described in this review focus mainly on numerosities above the subitizing range.

Unlike Feigenson et al. (2004), Cantlon et al. (2009b) do not distinguish between the representation of small and large numerosities. Instead, they suggest that different magnitudes, both numerical and continuous, share common representation mechanisms (see Fig. 1C),

specifically that numerosities, as well as other magnitudes, are represented by an approximate number system (ANS) (for a similar view, see Cohen Kadosh & Walsh 2008; Cohen Kadosh et al. 2008). The first evidence supporting shared representation is that perception of different continuums, like numerosity, space, and loudness of pitch, follows Weber's law. In addition, several studies indicate that other than for numbers, the parietal lobe, and specifically the intraparietal sulcus (IPS), is activated on estimations of continuous magnitudes. Cantlon et al. (2009b, p. 89) propose that "a system that once computed one magnitude (e.g., size) could have been hijacked to perform judgments along a new dimension (e.g., number)" (see also Henik et al. 2012). It is not yet clear whether neurons in the parietal lobe are general magnitude neurons that are activated in all magnitude judgments, or whether there are specialized neurons for different magnitudes and they are all intermixed in the same area (for further discussion, see Leibovich & Ansari 2016).

The theories mentioned previously emphasize the existence of a built-in module that can "sense" numerosities. There are, however, some theoretical concerns regarding these theories. Mix et al. (2002a) claim that the evidence showing that infants can process numerosities is ambiguous. When comparing two sets of stimuli of different numerosities, in addition to numerosity there are differences in overall contour length, total surface area, and so forth. These continuous magnitudes can serve as an alternative explanation to a number sense. Previous studies have referred to these confounds and tried to break the correlation between numerosity and continuous magnitudes. However, a re-examination of the literature indicates that the influence of continuous magnitudes on performance in numerosity comparison tasks cannot be ruled out

(Gebuis & Reynvoet 2012a; 2012b; Gebuis et al. 2014; Hurewitz et al. 2006; Leibovich & Ansari 2016; Leibovich & Henik 2013; Soltész & Szücs 2014). We now review the evidence supporting the number sense view and discuss its validity in the light of theoretical and methodological considerations.

3. Re-examination of behavioral evidence for the number sense

Non-symbolic stimuli are useful when studying basic numerosity processing; they can be used with different populations – adults, infants, and animals. Non-symbolic stimuli, however, also contain non-numerical continuous magnitudes. Because of the potential influence of these continuous magnitudes and because it is impossible to create two sets of items that differ in numerosity only (Leibovich & Henik 2013; see also Fig. 2), various studies have employed different methods to try to minimize the effect of continuous magnitudes in numerosity comparison tasks. Three main methods are used to do so. The first method is to manipulate one of the continuous magnitudes (i.e., keep one continuous magnitude constant across different numerosities) or to manipulate one continuous magnitude so it is not correlated with numerosity. The second method is to manipulate different continuous magnitudes in each trial so that in a given stimulus, only one magnitude is manipulated, but throughout the experiment, several magnitudes are manipulated. The third method uses different congruency conditions between numerosity and continuous magnitudes. We now demonstrate the different methods and show why it is still possible that continuous magnitudes affect participant performance.

The first method is based on the logic that continuous magnitudes are correlated with each other; therefore, it is sufficient to manipulate only one continuous magnitude. For example, as the average size of a set of stimuli increases, the total circumference of the dots, the total area they occupy, and so forth also increase. In the study of Abreu-Mendoza and Arias-Trejo (2015), participants

were presented with two arrays of items and were asked to decide which array had more items. To prevent participants from using continuous magnitudes in making their decision, the total surface areas of the arrays were equal. It is impossible, however, to change one continuous magnitude (total area in this case) without changing the others. In this example, when equating the areas of two arrays with different numerosities, the less numerous array has a greater average diameter necessarily. Hence, participants can rely on average diameter when making a decision. Accordingly, in this study, even though it is reasonable to assume that the numerical decision was not based on area, it is still possible it was made using other continuous magnitudes and not necessarily numerosity (for more studies that used a similar method, see Chassy & Grodd 2012; Im et al. 2016; Mussolin et al. 2010).

In the second method, several continuous magnitudes are manipulated throughout the experiment, but in a given stimulus, only one magnitude is manipulated. For example, Halberda et al. (2008) asked 14-year-olds to decide which of two colored dots were more numerous in a single array of yellow and blue dots. In half of the trials, the average size of the blue dots was equal to the average size of yellow dots. In the other half, the total areas of blue and yellow dots were identical. As in the first method, in a given trial, participants can use the other continuous magnitudes that were not manipulated. For example, in trials in which the areas of the dots were equal, the more numerous dots were smaller. These other continuous magnitudes can be used as a predictor of numerosity. Empirical data demonstrate this weakness; Tokita and Ishiguchi (2010) asked adult participants to compare target numerosity with standard (5, 10, 20, or 40) numerosity. In a third of the trials, the areas of individual dots were equated; in another third, the total area of the arrays were equated; and in the remaining third, both were equated. The Weber fraction and the point of subjective equality (PSE; i.e., the point at which the perceived numerosities of target and standard stimuli seem equal) were calculated for all conditions. Participants underestimated numerosity of large elements and overestimated
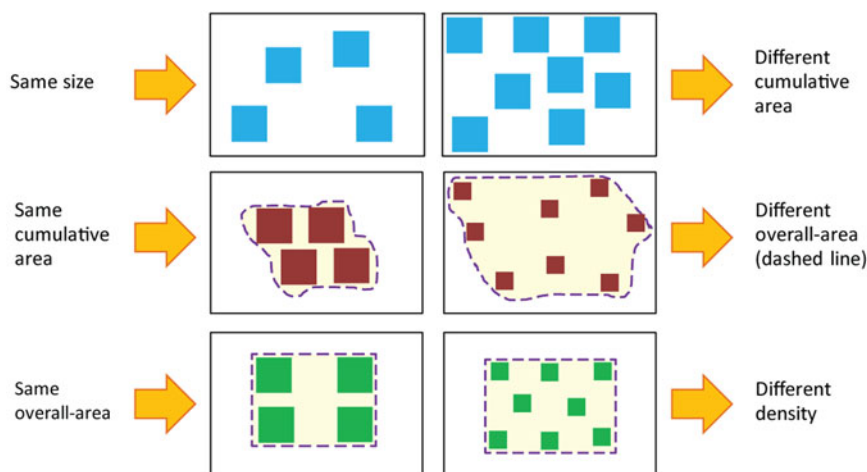

Figure 2. Correlation between number and continuous magnitudes. As illustrated in the figure, an attempt to equate one continuous magnitude in two different groups of items changes other continuous magnitudes, so that it is virtually impossible to get two groups of items that will vary only in their numerosity. Reprinted from Leibovich and Henik (2013).

numerosity of small elements. Namely, continuous magnitudes affected performance even when they were manipulated differently in different trials (for more studies that use this method, see Barth et al. 2005; Eger et al. 2015; Fazio et al. 2014; Gomez et al. 2015; Mussolin et al. 2010).

The third method for manipulating continuous magnitudes is to employ different congruency conditions between numerosity and continuous magnitudes. For example, in the study of Nys and Content (2012), adult participants performed a numerical or area comparison task. The stimuli were composed of dot arrays and different congruency conditions: congruent, meaning that the more numerous array had more area than the less numerous array; and incongruent, meaning that the less numerous array had more area. An interaction was found between task and congruency. Namely, the difference in performance between congruent and incongruent trials (i.e., congruency effect) was greater in the area comparison task than in the number comparison task. In other words, number interfered more when it was irrelevant, compared with area. Accordingly, it was concluded that numerosity is a more salient cue than continuous magnitudes (for other studies using congruency, see Barth et al. 2005; Bonny & Lourenco 2013; Nys & Content 2012). There are, however, contradictory findings in the literature (Durgin 2008; Gebuis & Reynvoet 2012b; Leibovich et al. 2015;

2016a; Szücs et al. 2013). Hurewitz et al. (2006) conducted a similar study and found that numbers affected performance in the area comparison task only when the numerical ratio was closer to zero (i.e., very large differences in numerosity). They reached the opposite conclusion from Nys and Content and argued that area is a more salient cue than numerosity. Recently, Leibovich et al. (2015; 2016a) found that number interfered with performance in a non-symbolic comparison task only if it was prompted by being the relevant dimension in a previous task.

A very strong line of evidence supporting the ANS, and especially the claim that number sense is innate, comes from cross-modal matching tasks with infants and animals. In such tasks, subjects are exposed to visual and auditory displays of a number of objects or events (see Fig. 3D); for example, a visual display of two and three dots and the sound of three tones. It has been found that subjects prefer to look at the visual display that matches the number of tones, that is, two objects when hearing two tones and three objects when hearing three tones (e.g., Jordan & Baker 2011; Jordan et al. 2008b). Such evidence, however, should be taken with a grain of salt. As reviewed by Cantrell and Smith (2013), with 5- to 8-month-old infants as participants, only two of six studies reported preferred matching between auditory and visual quantities (see their Table 7). In studies with newborns
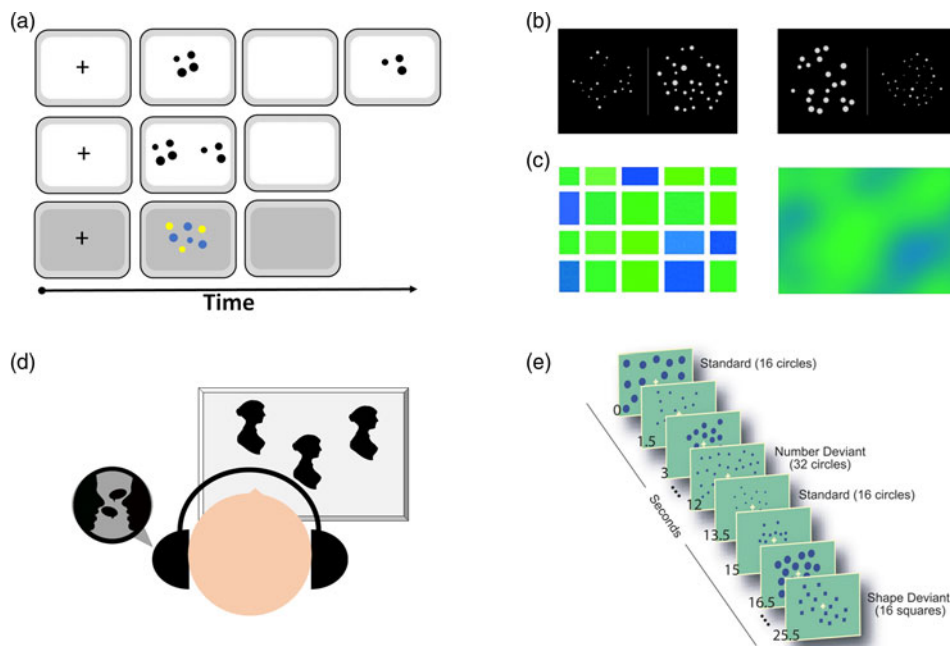

Figure 3. Examples of tasks for human participants. (A) Comparison task: two stimuli are shown side by side sequentially or simultaneously. Sometimes, one array with two colored dots is used (Agrillo et al. 2015). (B) Non-symbolic Stroop task (based on Leibovich et al. 2015; 2016b). In such a task, participants are asked to choose either the group containing more dots (e.g., number comparison task) or the group containing more area (e.g., area comparison task). All five continuous magnitudes (density, area, convex hull, etc.) are either congruent or incongruent with numerosity. (C) Castelli et al.'s (2006) task, in which participants need to indicate whether they see more blue or more green. The left stimulus aims to study numerosity processing, whereas the right stimulus aims to study processing of continuous magnitudes. It is clear from the left panel that the answer could be reached by both numerosity processing and continuous magnitude processing. (D) Cross-modal matching task, in which subjects are exposed to a number of visual and auditory stimuli. Looking time is longer when the numbers match than when the numbers do not match. (E) Passive habituation task (Cantlon et al. 2006), in which participants are repeatedly presented with a specific numerosity. In the dishabituation phase, the number changes. To account for general detection of change, a control condition is employed in which the shape of the items, but not their number, changes.

as participants (e.g., Izard et al. 2009), because of poor visual acuity, they are unlikely to be able to see objects that are placed relatively close to one another as being separate from one another, and they lack the ability to separate between object and background or between one object and another. Therefore, it is hard to interpret the results as an indication of an innate number sense. Another possibility might be that when participants hear "more," they prefer to look at "more" (Mix et al. 2016). With that being said, pinpointing the exact conditions under which cross-modal matching is observed can be very informative to further understand magnitude processing in infancy. Similarly, cross-modal priming studies were previously taken as evidence for a generalized magnitude system and not necessarily as evidence for the existence of an approximate number system (e.g., Oppenheimer et al. 2008).

Another line of evidence for the innate nature of the number sense comes from habituation studies with newborns and infants. In habituation studies, subjects are serially exposed to the same type of stimuli that have a common feature (e.g., the same number of items). This is done until subjects' looking time decreases. The reduction in looking time is taken as a sign that the common feature has been detected and the stimuli have lost their novelty. In the test phase, the subjects are exposed to a new stimulus (e.g., a different number of items), and the looking time is measured. When a novelty has been detected, looking time increases. To ask whether infants are born with a number sense, such habituation studies included displaying the same number of items in different spatial locations and changing the number of items in the test phase. Because the looking time of infants increased in the test phase, the conclusion was that infants are able to detect change in quantities from birth. However, as mentioned previously, changing the number of items necessarily changes continuous magnitudes. Hence, the increase in looking time can occur in response to detecting a change in different continuous magnitudes (for a detailed review, see Mix et al. 2002a). Some works aimed to pit number against continuous magnitudes in a preferential looking time task. For example, Libertus et al. (2014) used dot arrays and changed the number of dots, keeping total surface area constant, or changed the total surface area, keeping the number constant. The authors concluded that infants preferred looking at changes in number compared with changes in total surface area. However, keeping total surface area constant does not mean that the dishabituation was to number. It could have been, for example, to the physical size of each dot, as physical size correlated with number. Because there is no way of confirming what infants habituated to, it is equally possible that these results demonstrate that "physical size of dots trumps cumulative area" (Libertus et al. 2014, p. 108).

To conclude, in this section we reviewed different methods that tried to minimize the effect of continuous magnitudes in numerosity comparison tasks and habituation tasks. These methods were based on the assumption that if continuous magnitudes were not relevant and not correlated with numerosity, they would not influence performance while comparing numerosities. Some empirical studies, however, show that even under these conditions, continuous magnitudes affect numerical estimations (Cantrell et al. 2015b; Gebuis & Reynvoet 2012a). We review evidence for the role of continuous magnitude in number processing at length in section 6.

4. The origin of the number sense: Evidence from animal studies

As stated earlier, the ability to compare magnitudes is not unique to humans; it is an important ability for survival across species. Therefore, it has been argued that the number sense is not specific to humans, but is shared across species. Put differently, not only humans, but also animals are born with the ability to process numerosities (e.g., Dehaene 1997; Feigenson et al. 2004; Nieder 2005).

In this context, the numerical abilities of different animals were tested – from primates to insects. Cantlon and Brannon (2006) trained rhesus monkeys to discriminate a range of 1–9 dots per array and then tested their ability to discriminate numerosities in the range of 10–30. The authors reported that the monkeys were able to extend the numerical rule to the higher numerosity range. Moreover, response time (RT) and accuracy patterns of the monkeys were similar to those of humans in the same task; RT and error rates of both monkeys and humans increased with an increase in the similarity between the numerosities of the dots (i.e., as the numerical ratio got closer to 1). Similar findings were also reported by Beran (2007). In a more recent study, Viswanathan and Nieder (2013) searched for the underlying brain circuitry supporting numerosity representation; monkeys performed a color discrimination task of dot arrays while activity from neurons in the ventral intraparietal sulcus and the dorsolateral prefrontal cortex was recorded. The monkeys were not trained on numerosity discrimination. The authors reported neuronal activity in the tested frontal and parietal areas that was tuned to specific numerosity; different neurons showed maximal firing rates in response to different numerosities. Taken together, these studies provide evidence supporting the existence of a spontaneous number sense, enabled by frontoparietal networks in the monkey's brain.

The number sense is not restricted to the visual modality. Sometimes other senses are used to estimate magnitudes. This was demonstrated in different mammals. Meck and Church (1983) revealed that rats can discriminate between durations and between different numerosities; rats were trained to choose between pressing on one of two levers according to a series of light flashes – one lever was associated with a longer duration, and one was associated with a shorter duration of the flashing light (see Fig. 4A). The rats were able to choose the right lever even when bursts of noise were used instead of light flashes. A more ecological example comes from the study of McComb et al. (1994). This study tested the ability of lions to discriminate numerosity in the auditory modality. Specifically, McComb et al. tested whether lions (in nature) can estimate the size of an opponent group based on the number of roaring sounds. The authors used playback of one or three roars to simulate the presence of an unfamiliar intruder. According to the results, the lions were able to discriminate the number of roars and act accordingly – to take flight if the number of roars indicated the presence of more numerous lions and to stay to fight if
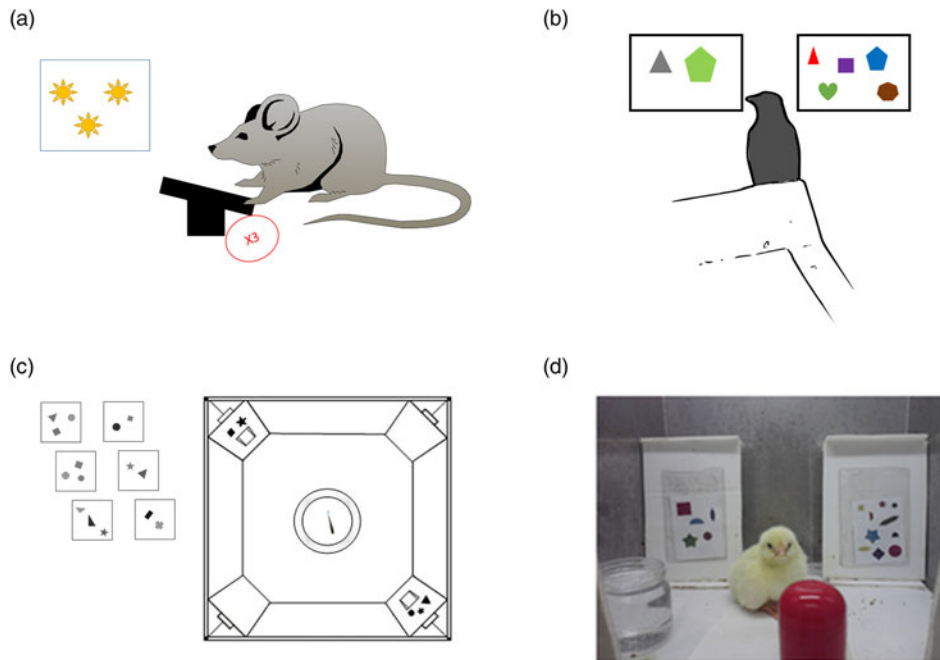
Figure 4. Examples of tasks for animals. (A) Illustration based on Meck and Church (1983). A rat sees a number of light flashes or hears a number of tones and needs to press a lever a similar number of times to get food. (B) Birds: A jungle crow is presented with two groups of items, one containing two items and one containing five items, and is trained to peck five items. In the test phase, novel quantities are presented (Bogale et al. 2011). (C) Fish: Fish are trained to go through the door with the larger number of items painted above it (Agrillo et al. 2009). (D) Chicks: Chicks are trained to select the screen with the larger number of items (Rugani et al. 2013).

the number of roars indicated the presence of a group with fewer lions.

To test how far back the number sense goes in evolution, studies were conducted with birds and fish whose brain structures differ from those of mammals and specifically primates. In these animals too, there is evidence supporting the existence of an innate number sense: Watanabe (1998) trained pigeons to respond to four objects and not respond to two objects. During the test trials, the pigeons responded to three, four, and five but not to two objects. Accordingly, the author suggested that pigeons could discriminate numerosities. Discrimination with even larger numerosities was reported in jungle crows by Bogale et al. (2011). In this work, jungle crows were trained to discriminate two from five. The crows also received a control test for non-numerical cues, such as spatial arrangement, shape, and total area. In the test phase, the crows showed the ability to discriminate between novel quantities, such as five and eight (see Fig. 4B). This discrimination, according to the authors, was not controlled by continuous magnitudes. Hence, the authors argued that much like other animals, jungle crows have a natural tendency to select the larger quantity and that this decision is affected by numerical ratio and stimuli magnitude. In an electrophysiological study by Ditz and Nieder (2015), crows were presented sequentially with two dot arrays and had to peck the second display of dots only if both displays contained the same numerosity. To minimize the influence of continuous magnitudes on performance, either surface area, total circumference, or the density of the dots was manipulated in different stimuli. Despite the very different neuroanatomy of birds, it was found that neurons in the endbrain of the bird (an area termed *nidopallium caudolaterale*

[NCL]) were tuned to a preferred numerosity. Fish, too, were shown to rely on numerosity to survive: Larger groups of fish (i.e., shoals) have less chance of being attacked by predators. For this reason, a fish that successfully chooses to join a larger shoal increases its chances of survival. This was demonstrated in a study by Agrillo et al. (2008). In this study, a single fish (mosquitofish) was placed in a central fish tank. This fish tank was flanked with two other fish tanks with different numbers of fish inside. It was found that the single fish spent more time next to the fish tank containing the larger number of fish. A later study by the same group (Agrillo et al. 2009) demonstrated that mosquitofish are able to discriminate between two and three objects even "when denied access to non-numerical information" (p. 1). In this study, fish were trained to discriminate between two and three sets of geometrical objects that varied in shape, size, brightness, and viewing distance. During the test phase, the fish were tested while controlling for one continuous magnitude at a time (see Fig. 4C). The authors reported that fish were able to discriminate two from three and that total luminance and the sum of the perimeters of the stimuli did not affect performance.

In all of these studies, however, as with humans, it was still impossible to control all of the continuous magnitudes and support claim that magnitude comparisons are based solely on numerosity judgments. For example, Cantlon and Brannon (2006) had three different types of dot arrays. In a third of the stimuli, the densities of the two dot arrays were equal. This means that the convex hull (the area occupied by all of the dots and the area surrounding them) was larger in the array with the larger numerosity. In another third, the surface areas of the two dot arrays

were equal. This means that the more numerous dots were smaller. In the remaining third of the dot arrays, the convex hulls of the two dot arrays were equal. This means that the more numerous dot array was denser. A similar approach was taken by Ditz and Nieder (2015). In some studies, only one or two continuous magnitudes were manipulated. For example, Bogale et al. (2011) controlled for total surface area but did not report other magnitudes such as density and convex hull. Similarly, Agrillo et al. (2009) reported controlling for item size and brightness, but not other continuous magnitudes that might have influenced performance.

The inability to keep all dimensions apart from numerosity constant is true for various modalities. In the study of Meck and Church (1983), increasing the number of flashing light events during 2 seconds meant that the tempo would be faster than when fewer flashing light events occurred in the same duration. In the study of McComb et al. (1994), increasing the number of roars affected the loudness of the roars.

As in the case for studies employing non-symbolic stimuli with humans at different stages of development, the most common method to rule out the influence of continuous magnitudes is to manipulate one continuous magnitude in a given stimulus so that overall, throughout the experiment, none of the continuous magnitudes can be used as a reliable cue of numerosity. The assumption is that under these conditions, participants will not use continuous magnitudes but will base their decision only on numerosity. As we will see, however, a growing body of evidence suggests that this assumption is wrong. In fact, participants are able to use continuous magnitudes even when they are irrelevant to the task and are not a reliable cue of numerosity (Gebuis & Reynvoet 2012a; 2013; Leibovich et al. 2015; 2016b).

5. Neural correlates of non-symbolic numerosities

The number sense theory assumes the existence of a "number detector," or specific brain tissue in the parietal lobe that is dedicated to the processing of numerosity (Piazza et al. 2010). Neuroimaging studies seeking such brain tissue have used stimuli similar to those used in behavioral studies (e.g., arrays of items). This poses a problem because different ways of manipulating continuous magnitudes might result in the activity of different brain regions (or different levels of activity of the same brain regions). Studies attempting to find brain areas dedicated to the processing of numerosity have used either comparison tasks (e.g., see Fig. 3A) or passive-viewing (habituation) tasks (Fig. 3E). We now review some examples of such studies and demonstrate the difficulty in attributing activity found for such tasks to pure non-symbolic numerosity processing.

In a functional magnetic resonance imaging (fMRI) study, participants compared either the numerosities of two presented dot arrays (i.e., number comparison task) or the physical sizes of two presented disks (i.e., size comparison task; Chassy & Grodd 2012). The right IPS was activated in both tasks. The contrast between the tasks revealed that the right superior parietal lobule (SPL) was more active in the number comparison than in the size comparison task, which according to the authors indicates that the SPL is involved in comparison of exact quantities.

Importantly, all of the dots in the number comparison task were presented in the same size. Thus, the total area was perfectly correlated with numerosity. Hence, it is possible that instead of comparing numerosity processing with size processing, the study compared continuous magnitude processing, which requires summation of the area of all of the dots (i.e., total dot area), with size processing, which does not require such summation (i.e., comparison of disk areas), and the areas found when contrasting activity in area and dot comparison tasks might reflect this difference rather than a difference in numerosity.

In the study of Cantlon et al. (2009a), 6- to 7-year-old children and adults performed a non-symbolic numerosity comparison task. Some of the continuous magnitudes were manipulated to prevent participants from using them as indicators of numerosity. The manipulation included presenting large numerosities with small dot size and small numerosities with large dot size, while keeping the same density for all trials. The left SPL was active in both children and adults. Although the brain activity in the study was attributed to numerosity processing, another way to interpret these results is by considering the consistent correlation between dot size and numerosity. By making the more numerous dots consistently smaller than the less numerous dots, participants could theoretically have responded according to the size of the dots and not their numerosity. In addition, keeping density constant creates a correlation between numerosity and convex hull. Namely, convex hull increases with numerosity and therefore could have been used as an indicator of numerosity.

In the study of Holloway et al. (2010), a different continuous magnitude manipulation was applied; adult participants were presented with two arrays of squares and were asked to choose the display side containing more squares. In these arrays, the individual size of each square varied. To prevent participants from relying on density and total surface area, these continuous magnitudes were manipulated so that in 25% of the trials, both magnitudes were congruent with numerosity (i.e., the more numerous squares were also denser, and their overall area was greater than the other array); in 25% of trials, both density and total surface area were incongruent with numerosity; and in the other 50% of trials, only density (25%) or total surface area (25%) was congruent with numerosity. Brain activity in this task was contrasted with a control condition in which the same stimuli were combined into a single irregular shape and participants were asked to decide which shape more resembled a diagonal line. The right inferior parietal lobule (IPL) and right SPL were found to be more active in the numerosity task compared with the control task. These areas are probably involved in numerosity comparison, but is that the only explanation? In addition to its role in numerical cognition, the IPL is also involved in cognitive control; IPL activity increases with conflict (Brass et al. 2005; Greene et al. 2004). Accordingly, activity of this area can also reflect different levels of conflict in the two tasks.

Harvey et al. (2013) manipulated continuous magnitudes by including several control conditions. In each condition, a different continuous magnitude was held constant. For example, density was held constant while numerosity changed (i.e., a small number of dots were presented using a large dot size, and a large number of dots were presented using a small dot size to create the same density level for different numerosities). The task included a

display of a dot array that could appear in either black or white (in different trials). Adult participants were asked to indicate the color of the dots in the array. It was found that as numerosity increased, brain activity shifted from medial to lateral areas of the posterior SPL. The interpretation given by Harvey et al. indicates that the topographic organization evidence was in line with previous reports of the role of the SPL in numerosity manipulations (Cantlon et al. 2009a). However, Gebuis et al. (2014) claimed that it is arguable whether the neural processing was directed at pure numerosity without the confounding continuous magnitudes. Gebuis et al. pointed out that the topographic representation found in each condition actually encoded the changes in the continuous magnitudes that were not held constant in each condition. The positive relationship between numerosity and continuous magnitudes, taken together with the parietal lobe involvement in sensory integration (Calton & Taube 2009; Naghavi & Nyberg 2005; Shafritz et al. 2002; Vallentin & Nieder 2010; Walsh 2003), led Gebuis et al. to propose that the suggested topographic map "reflects a weighted response of neurons that encode different sensory cues rather than a pure numerosity estimate" (p. 1).

Castelli et al.'s (2006) study is the only study, to the best of our knowledge, to directly compare brain areas that were active during numerosity and continuous magnitude comparison tasks. In the continuous condition, the stimulus was a rectangle containing alternating shades of green and blue. In the numerosity condition, the rectangle was composed of individual (and therefore countable) squares of blue and green rectangles. In both tasks, participants had to choose whether they were presented with "more blue or more green" (see Fig. 3C). The authors reported that bilateral regions of the IPS and regions of the parietal–occipital transition zone were significantly more active during the processing of numerosities than durig the processing of continuous magnitudes. Therefore, the results of this study were taken as evidence that the IPS is more strongly activated when numerical magnitudes are processed than when continuous magnitudes are processed. Note, however, that although in the continuous task participants were able to judge whether they saw more blue or more green only by using continuous magnitudes (e.g., area), in the numerosity task, participants were also able to base their decisions on comparison of the area covered with blue to the area covered with green. Hence, brain regions found in this contrast are not necessarily specific to numerosities.

5.1. Habituation studies at the brain level

Another way of revealing areas that are specific to numerosity processing is by using habituation studies. The idea behind such studies is to repeatedly present the same numerosity (i.e., habituation) and change all other magnitudes to find areas that will be more active when a new numerosity is eventually presented (i.e., during dishabituation). Such brain areas are assumed to be involved with the processing of numerosity. Cantlon et al. (2006) hypothesized that non-symbolic numerosity would activate the IPS in both 4-year-old children and adults. In this study, participants were repeatedly presented with 16 dots that differed in density, cumulative area, and spatial arrangement. Then, a novel stimulus was presented. In the novel

stimulus, either the number of dots changed (to 32) or the shape of the dots changed (to triangles or squares; see Fig. 3E). The IPS response to novel stimuli that differed in number versus the response to novel stimuli that differed in shape was measured. Results indicated that both 4-year-old children and adults activated the IPS (bilaterally) in response to a change in number and not shape; therefore, the authors claimed that the IPS is specific to numerosity processing. This study has two limitations. The first one concerns the manipulation of only three continuous magnitudes. In a habituation study, it is important to prevent habituation effects to irrelevant properties by presenting a variety of exemplars. This principle was applied to only three continuous magnitudes, leaving all other possible magnitudes exposed to habituation. If there is at least one continuous magnitude that correlated with numerosity and was not varied consistently (total circumference, for example), the habituation is no longer specific to numerosity. The second limitation concerns the contrast between the changes in number versus the changes in shape during dishabituation. The stimuli changed by a ratio of 2:1 and from dots to squares or rectangles. In a habituation design (without an active task to reflect saliency), it is impossible to be sure that changing the number of dots from 16 to 32 has the same saliency as changing a circle to a square. Therefore, the differences in IPS activation might reflect differences in saliency.

As far as we know, there is currently only one fMRI study that was able to show a distance-dependent dishabituation of a specific brain region to a change in non-symbolic numerosity. Piazza et al. (2004) habituated participants to a specific numerosity. The numerosities used for dishabituation varied in numerical distance from the habituated numerosity. The results revealed that the activity in the IPS region (and surrounding areas) during dishabituation correlated with the numerical distance; activity was higher for large numerical distance and smaller for small numerical distance. Accordingly, the authors suggested that numerosities are extracted automatically from a visual scene. Gebuis and Reynvoet (2012a), however, argued that it is possible that participants in the study of Piazza et al. integrated multiple continuous magnitudes presented to them, because not all of the continuous magnitudes were discorrelated with numerosity.

So far we have reviewed empirical evidence supporting the claim that numerosity processing is basic, innate, and automatic (i.e., supporting the number sense theory). Specifically, we have reviewed behavioral and neuroimaging studies with animals, infants, children, and adults, concluding that numerosity processing is evolutionarily ancient, innate, and automatic; this processing takes place in dedicated brain circuitries. All of these studies share the assumption that if a continuous magnitude is not correlated with numerosity, it will not be processed and will not affect performance. Unfortunately, as mentioned before, it is impossible to avoid a correlation between numerosity and all of the continuous magnitudes at once, nor it is possible to present different numerosities with the same continuous magnitudes. Therefore, it is very possible that in the studies discussed previously, performance was affected, at least partially, by continuous magnitudes. In the next section, we review studies that directly tested the role of continuous magnitudes in numerosity comparison tasks and suggested that

when making numerical comparisons, we use both numerosity and continuous magnitudes.

6. Evidence supporting holistic processing of numerosity and continuous magnitudes

To evaluate the contribution of continuous magnitudes to numerosity comparisons, adult participants compared dot arrays containing 5–25 dots each (Leibovich & Henik 2014). In these dot arrays, all continuous magnitudes were manipulated at once, to be minimally correlated with numerosity, so they could not be used as a reliable cue for numerosity (Gebuis & Reynvoet 2011). The numerical ratio between the compared arrays and the ratio between five different continuous magnitudes were used as predictors of accuracy and RT. For both of these measures, the ratio between different continuous magnitudes explained about half of the explained variance. Namely, continuous magnitudes affected performance even though they were neither relevant to the task nor predictive of numerosity. This point has been recently demonstrated even in the subitizing range. A work by Salti et al. (2017) revealed that different manipulations of continuous magnitudes influence performance in a non-symbolic Stroop-like task with numerosities in the subitizing range (e.g., 2–4; for an example of a non-symbolic Stroop-like stimuli, see Fig. 3B).

Recent studies also suggested that numerosity processing might not be as automatic as previously assumed. Gebuis and Reynvoet (2013) employed a habituation paradigm in which participants were adapted to specific numerosities or to specific continuous magnitudes, while event-related potentials (ERPs) were recorded. The results were analyzed twice; one analysis considered the change in numerosity to be the dishabituation event. This analysis did not reveal any brain area that detected the change. In the second analysis, however, the change in continuous magnitudes was considered to be the dishabituation event. This analysis revealed brain areas that responded to the change in continuous magnitudes. The same pattern of results was repeated even when participants were told that the numerosity of the dots would change. In the light of these results, the authors suggested that it is continuous magnitudes, and not numerosities, that are being extracted automatically from arrays of items. Similarly, in another ERP study (Soltész & Szücs 2014), participants were habituated to either shape or number. Although shape adaptation occurred fast and in the range of early visual components, adaptation to number occurred only later. This pattern goes against the claim that numerosity processing is automatic like the processing of shapes and colors (Burr & Ross 2008). Recently, DeWind and colleagues (DeWind et al. 2015; Park et al. 2016b) used a new method for creating non-symbolic stimuli (groups of dots) and analyzing the influence of numerosity and of continuous magnitudes. They suggested that numerosity is processed automatically and very early in the visual stream. However, there are both methodological and empirical caveats concerning the method of DeWind and colleagues. First, in these works, it is not clear whether there is a correlation between number and continuous magnitudes, and if there is, how strong it is. This is important because a strong correlation between number and continuous magnitudes can change strategy. Namely, if, for example, the

correlation between area and numerosity is 0.7, then participants can reliably choose the larger area. The findings of DeWind and colleagues can also be accounted for by the *signal clarity* view, stating that numbers can be more salient in habituation studies simply because they have less variance than continuous magnitudes. For example, Park et al. (2016b) used five numerosities in their experiment. The variance in the continuous magnitudes was greater. This facilitates use of numerosity rather than other continuous magnitudes.

Recent studies also found that numerosity is not processed independently of continuous magnitudes, as proposed by the number sense theory. Gebuis and Reynvoet (2014) asked participants to passively view ordinal arrays of dots (e.g., groups of three, four, five, six, and nine dots) while ERPs were recorded. A trial was composed of five dot arrays, presented consecutively. Numerosity and five continuous magnitudes (convex hull, aggregate surface, density, diameter, and contour length) always increased with numerosity in the first four-dot arrays. Namely, more dots were denser, occupied more surface area, and so forth than fewer dots (i.e., congruent). In the fifth and last dot array, the continuous magnitudes were congruent in half of the trials or incongruent with numerosity in the other half. The authors found congruity-related changes in ERPs above left parietal and mid–right frontal electrodes. Specifically, these channels showed greater positive amplitude in response to incongruent trials than to congruent trials. This result cannot be reconciled with numerosity processing that is independent of continuous magnitudes.

Although, as demonstrated previously, it is very difficult (if not impossible) to isolate brain areas that are specific to processing of numerosity independent of continuous magnitudes, there is a strong line of evidence showing a great overlap between the processing of numerosity and continuous magnitudes (Cappelletti et al. 2013; Dormal et al. 2012). For example, in the fMRI study of Pinel et al. (2004), it was found that numerical size, physical size, and luminance activated bilateral IPS and occipitotemporal regions during comparison tasks. Behavioral analysis of the results revealed that both physical size and luminance affected performance in the numerical judgment task even when they were irrelevant. The authors argued that these results demonstrate distributed processing along the IPS, with some areas specific for one magnitude and a considerable overlap between all of them. Such evidence supports the shared representation of space, time, and number suggested previously by Walsh (2003).

The majority of non-symbolic comparison studies, including all imaging studies mentioned so far, examine numerosity above the subitizing range. One fMRI study that dealt specifically with performance in the subitizing range was conducted by Leibovich et al. (2015). The study manipulated the congruity of numerosity and continuous magnitudes of dot arrays (half of the trials in each task were congruent, and half were incongruent) and task order (half of the participants started with the numerosity discrimination task, and half with the area discrimination task). The results indicated that performance was faster and more accurate in the area task. Importantly, although area always affected performance in the numerosity task, numerosity affected performance in the area task only for participants who started with the numerosity discrimination task. Interestingly, the order in which the tasks were

administered affected activations at the brain level. Namely, during congruent trials, the group that started with the numerosity task showed activation in the right frontoparietal areas, whereas the group that started with the area task showed activation in homologue left frontoparietal areas. This study demonstrates that even in the subitizing range, continuous magnitudes still influence performance at both behavioral and functional levels, and this effect is further modulated by task context.

Further evidence supporting shared representation of different magnitudes derives from neural recording studies in primates. In the study of Tudusciuc and Nieder (2007), rhesus monkeys were presented with a sample stimulus (either a group of dots in the numerosity task or a line in the continuous task). After a short delay, a test stimulus appeared. The test stimulus matched the sample stimulus (in either numerosity or length) only in half of the trials. The monkeys had to respond only when the test stimulus matched the sample stimulus (i.e., a delayed match-to-sample task). During training, electrodes recorded the neuronal activity in the depth of the monkeys' IPS. This study revealed the existence of a neural population that was active for both the continuous and numerosity tasks, suggesting that some neural populations are involved in general magnitude processing. The authors speculated that this neuron population resides in an area equivalent to that found in humans by Pinel et al. (2004).

In addition to studies at the brain level presented previously, molecular studies with animals provide evidence for molecular mechanisms allowing size discrimination during early visual processing. In their seminal paper, Lettvin et al. (1959) noticed that a frog's choice of food is driven by size and motion. In their own words, "He [a frog] can be fooled easily not only by a bit of dangled meat but by *any* moving *small* object" (p. 1940). This observation implies a mechanism that can discriminate between small and large objects to choose the appropriate behavior. Such size-based decisions are important for other species too. Zebrafish, for example, behave differently in response to small and large objects; the zebrafish approaches small stimuli (having the size of its prey) and avoids larger stimuli (having the size of its predators). This pattern of results was replicated even when the stimuli were not animals, but small and large squares, further supporting the suggestion of Lettvin et al. that size alone is sufficient to modify behavior. Preuss et al. (2014) went further and asked which molecular mechanisms can explain such behavior. These authors provided direct evidence revealing that neurons in the deeper layer of the tectum (the uppermost part of the midbrain) provide inhibitory input to the tectum in response to small objects. In contrast, the superficial neurons in this area provide inhibitory output to the tectum, unless a large object comes into view (see also Abbas & Meyer 2014). These two studies suggest that the processing of size is very basic and innate. Hence, it is possible that size and other continuous magnitudes play an important role not only in evolutionarily older animals, but also in primates and particularly in humans.

7. The role of continuous magnitudes in numerical cognition

So far we have reviewed some studies that support the number sense theory and some that do not. One reason for these contradicting findings can be the result of different ways of manipulating continuous magnitudes. This possibility was tested recently by Smets et al. (2015; but see also Gebuis & Reynvoet 2012c). The authors employed several conditions to directly evaluate the influence of different methods of continuous magnitude manipulation on performance in non-symbolic estimation and comparison tasks. In the first condition, the surface area was kept constant in half of the trials and diameter was kept constant in the other half; in the second condition, four continuous magnitudes (convex hull, total surface, average diameter, and density) were manipulated at once so they were not correlated with the numerical ratio; and in the third condition, all continuous magnitudes were correlated with numerosities. Performance (measured using the Weber fraction, a measure based on accuracy that describes the smallest difference in numerosity that can still be discriminated) was modulated by the method of controlling continuous magnitudes; performance was worst when manipulating four continuous magnitudes simultaneously, better when only one continuous magnitude was manipulated, and best when all of the continuous magnitudes were correlated with numerosity. The differences in performance were more noticeable when the numerosities were more similar (i.e., having higher numerical ratios). A similar pattern of results was reported by Clayton et al. (2015). These studies reflect one of the major problems in studying non-symbolic numerosity comparisons – performance is highly sensitive to the way researchers choose to manipulate continuous magnitudes. The question then remains, what is really measured in such tasks? Is it the ability to process numerosity? Is it the ability to integrate multiple magnitudes to make a decision? Is it the ability to inhibit irrelevant continuous magnitudes?

Such questions led to a new line of theories suggesting that continuous magnitudes play an important role in the development of basic numerical abilities. This new line of theories takes into account the natural correlation between numerosity and continuous magnitudes and proposes a general system to process different magnitudes. The logic behind these theories is that any given group of items has, in addition to numerosity, different continuous magnitudes that usually correlate with numerosity. Thus, when required to make a numerical decision, continuous magnitudes will also be used. In other words, we use all available cues to make an informed decision rather than relying on a single magnitude. Accordingly, humans (and other species) will use any information available to make decisions. In this context, Gebuis and Reynvoet (2012b) suggested that developing a system unique for numbers is redundant and costly. Because numerosity and continuous magnitudes are usually positively correlated, it is more likely that we have a general magnitude system that makes numerical decisions according to both continuous magnitudes and numerosity, as suggested previously (e.g., Cantlon et al. 2009b). In other words, they propose a holistic process in which people do not extract numerosity independently from the other continuous magnitudes. Although the idea of a generalized magnitude system is not new and has been explored in multiple studies (e.g., Meck & Church 1983; Moyer & Landauer 1967), these studies suggest not only that number and non-numerical magnitudes are processed

by the same system, but also that the processing of non-numerical magnitudes is automatic and hence influences empirical efforts to measure the ability to compare or estimate processing of non-symbolic numerosities.

Leibovich and Henik (2013) acknowledged the challenges in assuming that numerosities are processed independently from continuous magnitudes and presented a developmental model for the acquisition of numerical knowledge. According to this model, people are not necessarily born with the ability to represent numerosities, but are able to develop this ability because of the correlation between numerosity and continuous magnitudes. Their theoretical model suggests that humans are born with the innate ability to distinguish between continuous magnitudes, not numerosities. With time, infants explore their environment and learn by experience that usually more toys will take more space in their toy box, or put more generally, that numerosity and continuous magnitudes positively correlate. It is only after understanding this correlation that the number sense and the idea of numerosity can develop (for a similar ideas see Henik et al. 2017).

Furthermore, a new and important theory, the signal clarity theory (Cantrell & Smith 2013), suggests that the influence of numerosity on performance can be a result of the experimental design. Specifically, this view claims that an important distinction should be made between what happens in the natural environment and in an experimental setup. In the environment, numerosity and continuous magnitudes usually correlate. In an experimental setup, in contrast, researchers control and manipulate continuous magnitudes by breaking this correlation. This manipulation creates greater variance of the continuous magnitudes and very small variance in numerosity. A possible outcome is that the numerical signal is much more salient than the continuous signals. For example, if in a habituation task six items are consecutively presented and all continuous magnitudes vary, it is much more likely that participants will pay more attention to changes in numerosities. According to the signal clarity view, infants are able to learn "on-line," during an experimental session, which dimension has the least variance and to use the information. Importantly, experiments do not use environmental scenarios; hence, the results obtained do not necessarily represent reality (Cantrell & Smith 2013).

8. New outlook on the contribution of cognitive control to the development of a number sense

The theories mentioned previously and others (e.g., Gebuis & Reynvoet 2012b; Gevers et al. 2016; Henik et al. 2017; Leibovich & Henik 2013; Mix et al. 2016) suggest that numerosities and continuous magnitudes are processed holistically because they usually correlate. This correlation, however, is not perfect. An everyday example of the violation of this correlation is the following scenario in a grocery store: When waiting in line, you will usually choose to wait behind the person with the emptiest cart because usually less-filled carts carry fewer items (Fig. 5A). There are, however, exceptions to this rule; you will wait less time after a full cart containing a few large items than after a half-empty cart containing many small items (Fig. 5B).
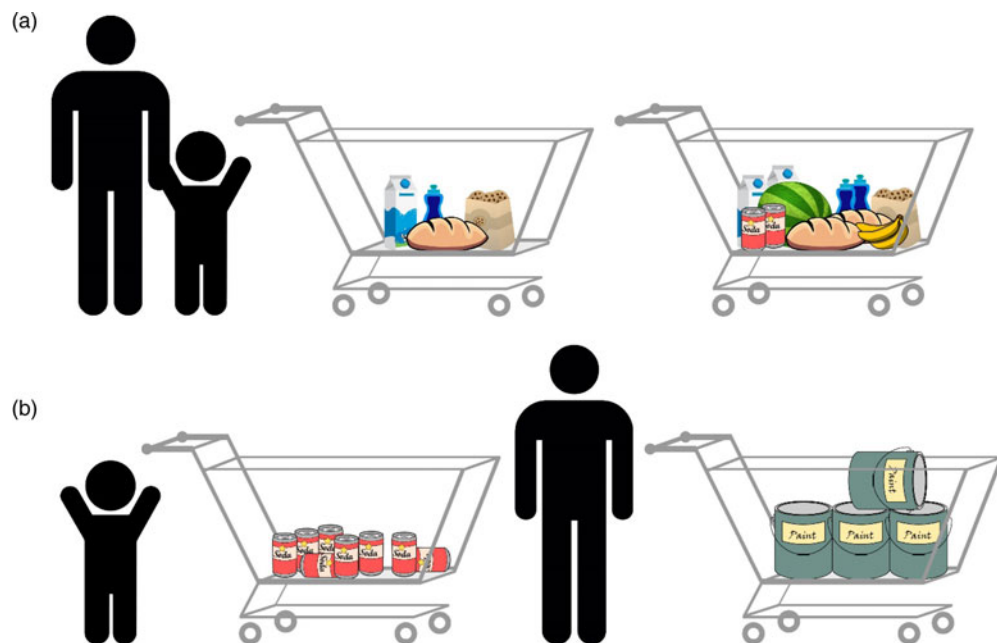

Figure 5. Numerosity comparison at the grocery store. (A) An example of a typical correlation: Usually, more items will occupy more space, will have greater surface area, and so forth in comparison with fewer items. In this case, understanding the correlation and using it facilitate performance, allowing us to make faster and more accurate decisions. Both adults and children are able to indicate which cart has more items. (B) Violation of this correlation: The full cart on the right contains fewer items, but they still take more space and have greater surface area than all of the items in the other cart. Although adults can identify the violation of the correlation and choose the fuller cart because of the ability to inhibit the influence of continuous variables, young children might not be able to understand that the full cart has fewer items.

Accordingly, it is possible that both integration and inhibition abilities, which are different aspects of cognitive control, are required to allow us to use the natural correlation between numerosities and continuous magnitudes, without being "enslaved" by it, namely, to be able to use this correlation (i.e., integration) and to deal with its violations (i.e., inhibition). It is well known that cognitive control abilities, including inhibition, develop with age (e.g., Morton 2010). It is also known that cognitive control correlates with math abilities (e.g., Bull et al. 2008). The exact role of cognitive control in the process of acquiring mathematical knowledge, however, is less clear because performance in different numerical comparison tasks does not always correlate with cognitive control abilities (Clayton & Gilmore 2015; Keller & Libertus 2015; Smets et al. 2015).

Additionally, in non-symbolic numerosity comparison tasks, it is virtually impossible to dissociate cognitive control from ANS acuity. The Weber fraction computed from a dot discrimination task is considered an indicator of ANS acuity. It has been demonstrated, though, that the Weber fraction is influenced by the level of congruity between numerosity and continuous magnitudes in both children and adults. Specifically, the Weber fraction was higher (i.e., performance was worse) in incongruent than in congruent trials (Tokita & Ishiguchi 2013). On the other hand, to avoid this confound means eliminating the conflict, that is, to use pairs of stimuli in which numerical and continuous magnitude perfectly correlate. However, in this case, one cannot be sure that participants are basing their decision on numerosity, continuous magnitude, or a combination of both. Hence, the ability to discriminate between very similar numerosities and the ability to inhibit irrelevant continuous magnitudes are inseparable. This not only is true in behavioral studies, but also might apply to neuroimaging methods; at the brain level, during incongruent trials it is reasonable to expect parietal areas to be active in response to processing numerosities, and frontal areas to be active in response to conflict. This, however, is an oversimplification because there are areas in the frontoparietal network that are involved in both cognitive control and processing of size (Brass et al. 2005), and frontal areas that are associated with cognitive control, such as the inferior frontal gyrus (IFG), are also reported consistently in numerosity comparison tasks (Kaufmann et al. 2005). Another example comes from a recent fMRI study by Leibovich et al. (2015; 2016b). In this study, participants compared the total area or the number of either congruent or incongruent dot arrays. IPS activity was found only when contrasting congruent versus incongruent trials, but not area versus dot comparison. Hence, IPS activity in this comparison task was probably related to conflict. It is possible that methods that correlate a specific pattern of activity of the same area in different tasks, such as multivariate voxel pattern analysis (MVPA), will eventually be able to partially separate ANS acuity from cognitive control, but as of now, such data do not exist.

Against this background, we put forward a theoretical model for the development of mathematical abilities. This model is similar to some recently suggested models (Mix et al. 2016) in that it includes the role of continuous magnitudes and assumes a general system for processing magnitudes. This model, however, also has some unique suggestions. First, we hypothesize that

number sense is not innate, but acquired. We emphasize the role of the correlation between number and continuous magnitudes in the process of learning the concept of numbers. We also take a more domain-general view and discuss the role of cognitive abilities such as individuation, language, and cognitive control in the development of the concept of number. This model expands two previously suggested models (Leibovich & Henik 2013; Leibovich et al. 2016a).

This model assumes that humans are not born with the ability to discriminate numerosities. Instead, babies use continuous magnitudes of groups of objects when comparing them. There are two reasons for this assumption. First, newborns have very poor visual acuity. Until the age of 4 weeks, babies are unable to focus their vision because muscles in the eye, fovea, and brain areas related to visual processing have not fully matured. The result is very blurry vision (Fig. 6). It is only at the age of 8 weeks that babies can focus on objects, but only at distances up to 100 cm (Banks 1980; Dobson & Teller 1978). The second reason is that babies are not born with the cognitive ability to individuate objects. In other words, until the age of about 5 months, babies cannot tell where one object ends and another one begins (Carey 2001). Without the ability to individuate, it is unclear how the concept of the number of items can even be applicable.

Instead of being born with a sense of numbers, we suggest that a number sense develops from understanding the correlation between numerosity and continuous magnitudes (Fig. 7). Such a process was referred to in the past as "statistical learning" (e.g., Frost et al. 2013), in the context of object individuation: When babies notice that some properties stay together, they understand that these properties belong to the same object. Statistical learning also helps with studying language – to understand that some combinations of sounds go together more frequently (for an elaborate discussion, see Mix & Sandhofer 2007). Therefore, it is possible that a similar process can serve in learning the natural correlation between number and continuous magnitudes. Once babies are able to individuate items, they are exposed to the natural correlation of numerosities and continuous magnitudes. For example, more candies will occupy more space on the plate. It is because of such experiences that children learn this correlation and even over-apply it by constantly integrating both numerosity and continuous magnitudes when estimating numerosities. For that critical stage to occur, first some notion of numerosity should exist. We hypothesize that it is through language, or more specifically through number words, that the discrete quantity of a set (i.e., numerosity) is emphasized. Please note that Mix et al. (2016) have recently suggested that number words orient attention toward numerosities. Over-applylication of such correlations was observed in Piaget's (1952) number conservation task. In such a task, a child will see two rows containing the same number of items (e.g., coins). The items in the two rows are equally distant from one another (i.e., the two rows have the same total surface area and convex hull). The experimenter then asks the child if the two rows contain the same or different numbers of coins. The child usually answers "same." Then, the experimenter changes the spacing of the items in one row (in front of the child), creating a larger convex hull in one row, and repeats the question. As the convex hull differences become larger,
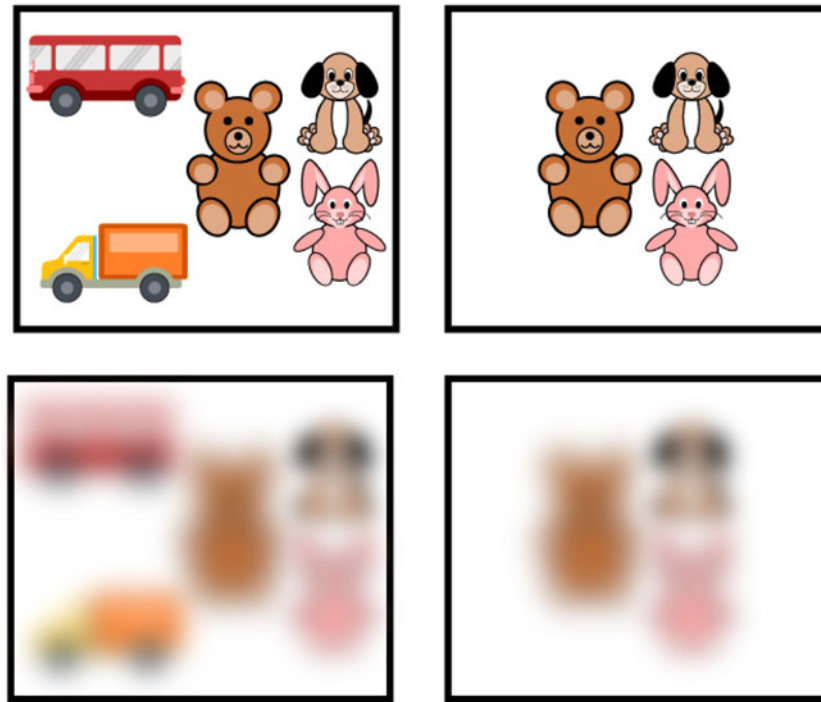
Figure 6. Poor visual acuity of newborns. Example of arrays of items and how they look with poor visual acuity. The images in the top squares demonstrate how groups of toys would look to an adult. The images in the bottom squares demonstrate how the groups of toys would look to newborns, whose visual acuity is 25 times worse than that of an adult, from a distance of more than 45 cm.

the child is more likely to indicate that the more spaced row contains more items. Older children will perform this task better; namely, they will be less confounded by the convex hull. More recently, it has been suggested that even though adults have high accuracy in such a task, they activate more cognitive control–related areas when the convex hull and the number of items are incongruent (Leroux et al. 2009).
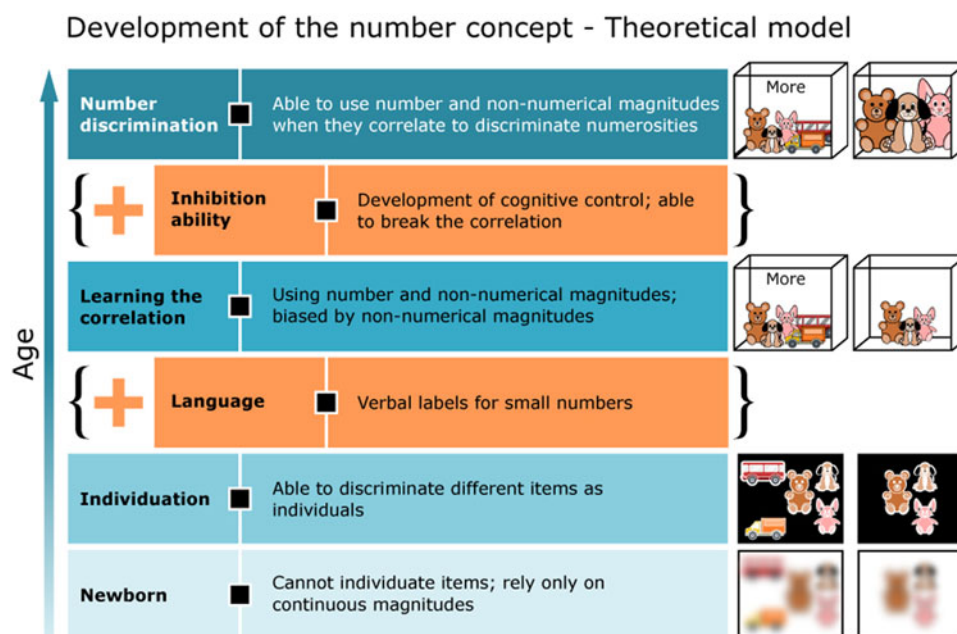

Figure 7. Theoretical model describing developmental landmarks of basic numerical abilities. Because of physiological constraints, newborns' vision is not acute enough to focus on specific items (Banks 1980). Only at the age of 5 months are babies able to individuate items from the background and from one another (Carey 2001). Individuation is critical to understanding the concept of numerosity. With the development of language and specifically number words, more attention is given to numerosities. With experience, a child learns correlations; for example, usually, more toys will take more space in the toy box. With the development of cognitive control and inhibition – general abilities that are related not only to math – a child can understand that correlations can be violated and compare numerosities even when they do not correlate with continuous variables. We suggest that this is the starting point required for basic math abilities.

To be able to deal with the violation of the correlation, proper cognitive control abilities should be developed. When inhibition abilities are well developed, it is possible to both use the correlation between numerosities and continuous magnitudes when it is appropriate and ignore continuous magnitudes when they are irrelevant, as in the case of the grocery store shopping carts mentioned earlier or when a child is asked to count the number of different-sized objects. It is only then that a child can really understand the concept of numbers and why, for example, the quantities of five apples and five watermelons are equal, even though watermelons are bigger. Count words may also play an important role in emphasizing numerosity over continuous magnitudes. As suggested by Mix et al. (2016), "Count words signal that number is a distinct property, independent of these other quantitative dimensions" (p. 20). In their work, they also review evidence suggesting that adults in Western cultures that do not have number words have problems discriminating or estimating large quantities.

9. Outstanding questions and future directions

9.1. Specifying the role of cognitive control in the development of basic numerical abilities

The role of cognitive control in the development of basic numerical abilities is not yet clear. One major challenge is that cognitive control is in fact a compilation of abilities. Hence, the first challenge is to find ways to "separate," if possible, numerical cognition abilities from cognitive control abilities. As discussed previously, it is virtually impossible to create a non-symbolic comparison task without creating the need to use cognitive control.

A second challenge stems from the fact that there is no single measure of cognitive control, and different studies that correlate math abilities with cognitive control test different cognitive control components. For example, one possible division relates to three major components: inhibition, updating, and shifting. These components consist of several sub-components (Miyake et al. 2000). Let us consider the case of inhibition. Two hallmark tasks to measure inhibition are the Stroop and flanker tasks. In the Stroop task, a color word is presented in a font color that is either congruent to the word (e.g., the word RED printed in red color) or incongruent (e.g., the word RED printed in green color). In the flanker task, subjects are asked to report the direction of the middle arrow in a horizontal line consisting of five arrows. Here too, the middle arrow can be congruent with the flanking arrows (e.g., all arrows are pointing right: →→→→→) or incongruent with the flanking arrows (e.g., the middle arrow is pointing left and the flanking arrows are pointing right: →→←→→). It has been demonstrated that performance in these two different tasks, both aimed at measuring inhibition, do not correlate (Shilling et al. 2002). To examine the role of cognitive control in the context of numerical abilities, different studies used different tasks to measure cognitive control, for example, the NEPSY (A Developmental Neuropsychological Assessment)-II inhibition subset, in which participants view a picture of a circle and need to say "square" (Gilmore et al. 2013; Keller & Libertus 2015); the Stroop task with words or numbers (Bull & Scerif 2001; St Clair-Thompson & Gathercole 2006); delayed

response (Espy et al. 2004); peg tapping (Blair & Razza 2007); and many more (for a review see Cragg & Gilmore 2014). Those different studies drew different conclusions regarding the correlation between math abilities and inhibition abilities, maybe because different tasks tap into different aspects of inhibition, and some aspects can be more related to math abilities than others.

It is important to understand the specific mechanisms of inhibition and cognitive control in general in numerical cognition. One way to shed light on this topic is to use a variety of tasks that measure cognitive control and examine their correlations with numerical abilities. Nevertheless, one should always be aware that correlation is not a causality, and direct empirical studies should be conducted to determine the underlying role of cognitive control in the development of numerical cognition.

9.2. The role of continuous magnitudes in dyscalculia

The reviewed body of work suggests that number sense may not be innate. Hence, it is important to ask how number sense develops in typically developed children and in children and adults with learning difficulties specific to mathematics (i.e., DD or MLD). DD and MLD manifest in different behaviors, which led Rubinsten and Henik (2009) to suggest that different brain dysfunctions may underlie each syndrome. The authors proposed a distinction between "pure" DD, comorbid DD (with attention deficit hyperactivity disorder [ADHD]/dyslexia), and MLD that arises from different brain dysfunctions. With respect to "pure" DD (which is attributed to dysfunction of the IPS), one possibility that comes to mind in this context is that difficulty in understanding the correlation between numerosity and continuous magnitudes can lead to impaired number sense and poor math abilities. This suggestion can be tested empirically. For example, by using a numerosity comparison task with different levels of congruency (Gebuis & Reynvoet 2012b), one can empirically test the ability of individuals with dyscalculia to rely on continuous magnitudes when they correlate with numerosity. Specifically, in typically developed adults, accuracy was found to increase when the number of continuous magnitudes that positively correlated with numerosity increased, suggesting that participants were able to use the correlation between continuous magnitudes and numerosities. If performance of individuals with dyscalculia does not have a similar pattern, it might mean that individuals with dyscalculia are not aware of the correlation between numerosities and continuous magnitudes. This would be a domain-specific account of dyscalculia (see also Gliksman et al. 2015).

There are, however, new studies suggesting a more domain-general account for dyscalculia. For example, in the study of Bugden and Ansari (2016), children with dyscalculia and typically developed children performed a non-symbolic comparison task in which the total area of the dots was either congruent or incongruent with numerosity. Although both groups performed similarly in the congruent trials, children with dyscalculia performed worse in the incongruent trials. Such a pattern suggests a more domain-general account of dyscalculia because the only difference between those with DD and typically developed children was in incongruent trials that required cognitive control. Such studies also demonstrate the importance of

understanding the specific role of cognitive control, facilitation, and inhibition in numerical cognition, as discussed previously.

9.3. Asking the right question: A neural base for holistic processing of numerosities

As demonstrated, it is virtually impossible to separate continuous magnitudes from numerosity processing, especially without triggering other cognitive processes such as cognitive control. Because number and continuous magnitudes cannot be separated, it is virtually impossible to empirically design a study that will show that *only* continuous magnitudes are processed or *only* numerosities are processed. Accordingly, at the brain level, it becomes difficult to define brain areas dedicated to numerosity processing. We would like to argue that in the light of compelling evidence for holistic processing of numerosity and continuous magnitudes, a suitable goal would be to ask which brain areas support integration of different dimensions of magnitudes (i.e., holistic processing). It has already been suggested by Walsh (2003) that space, time, and number are all being processed in the parietal lobe because integration of this information is needed to direct actions in the real world, for example, to know when to cross the road when a car is approaching. Indeed, areas in the parietal lobe, such as the IPS and the SPL, were found to be active during numerosity processing but also during tasks requiring integration of information (Graziano 2000; Jancke 2001). This integration might not be limited to these areas; for example, the right temporoparietal junction was found to be involved in feature integration. Specifically, right temporoparietal junction activity was found to be modulated by the number of congruent features presented to participants, even from multisensory inputs (Calvert et al. 2000; Pollmann et al. 2014). Pursuing a line of research that asks about integration of information throughout development and testing such integration abilities in individuals with different math abilities can broaden what we know about brain areas supporting numerical cognition and can help define which abilities should be improved to improve math abilities.

10. Conclusion

The main goal of this review is to encourage researchers not to assume that number sense is simply innate, but to put this hypothesis (almost regarded as an axiom) to the test. The theoretical model presented in this review raises more questions than answers. It is important to understand that the ANS theory and the suggested model are not mutually exclusive, and therefore, providing evidence against one theory does not necessarily mean that the other theory is right. This is especially true in the light of the methodological challenges of studying numerosity in isolation from continuous magnitudes. It is our hope that this review and our theoretical model will start a discussion that will result in new and exciting research directions aimed at investigating the possible role of continuous magnitudes and cognitive control abilities in typical and atypical development of math abilities. Such research directions can have crucial implications for the way we study behavioral and neural mechanisms relating to numerical cognition in humans and other species. Furthermore, these different directions can also lead to changes in the way basic math is taught and the way math difficulties are diagnosed and ameliorated.

NOTE
1. Naama Katzin and Maayan Harel contributed equally to this work.

Open Peer Commentary

The contribution of fish studies to the "number sense" debate

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Abstract: Leibovich et al. propose that number sense is not innate but gradually emergent during ontogeny following experience. We argue that this hypothesis cannot be reasonably tested in humans, in which the contribution of neural maturation and experience cannot be experimentally manipulated. Studies on animals, especially fish, can more effectively provide critical insights into the innate nature of numerical abilities.

Various authors have proposed that humans and animals integrate multiple magnitudes (number, area, density, etc.) when comparing numerosities; proposed mechanisms range from the idea that numerical information is more cognitively demanding than continuous magnitudes, and it is processed as a last-resort strategy when no other information is available, to the idea that the number system increases its precision by integrating available non-numerical information in the process of estimation and comparison (Agrillo et al. 2011; Davis & Perusse 1988; Meck & Church 1983).

What is new in the model proposed in the target article is the idea that humans and nonhuman species are born with a quantitative system that holistically processes numerosity and continuous magnitudes, and that a "sense of number" would gradually develop during ontogeny from understanding the correlation between numerosity and continuous magnitudes.

In the Introduction, the authors acknowledge the importance of animal studies for understanding the mechanisms of numerical discrimination. However, the evidence of such studies, whether in favor of or against their hypotheses, is not discussed. Here we argue that (1) experiments on animals, specifically fish research, can be more appropriate than research on humans to test some of the model's assumptions, especially to examine the hypothesis that number sense is not innate, but rather stems from individual experiences; and (2) evidence from animal studies that would be useful for evaluating the proposed model is already available.

Humans, other mammals, and most birds are extremely immature at birth, and the procedures commonly used to study number sense with adults (e.g., training procedures or free choice tests) cannot be employed; conversely, procedures used with young individuals (e.g., habituation or violation of expectancy) are usually complex to adapt to testing adults. This prevents researchers from comparing the different developmental stages with the same paradigm. Poor sensory acuity can further prevent testing for numerical abilities at very young ages (see Leibovich et al.). On the contrary, most fish species produce offspring that are completely independent at birth. Newly born fish generally face the same ecological challenges as adults (i.e., evading predators, selecting an appropriate diet and catching prey, orienting themselves in space, and interacting with conspecifics). This allows researchers exploiting a number of spontaneous behaviors, such as preferring the largest amount of food or the largest group of conspecifics, the same preferences studied in adults. Recently, we found that guppies can be trained to have numerical discrimination within their first week of life, which makes it a tool available for cross-age comparisons (Piffer et al. 2013).

Innate cognitive abilities often appear later in life, not because they need experience to develop, but because the maturation of the nervous system is required or because a given cognitive ability is not necessary for survival in early life (a fact that was not considered by Leibovich et al.). In these cases, answering the question of number sense innateness requires the manipulation of experience to disentangle the relative contribution of cerebral maturation and individual experiences on the development of numerical abilities. For both practical and ethical reasons, in higher vertebrates it is difficult, if not impossible, to devise experiments that dissociate the role of these two factors (e.g., LeVay et al. 1980; Ridley & Baker 1982). Such research is more feasible in fish. For example, one experiment found that in guppies that are prevented from experiencing different numbers of objects, the discrimination of large numerosities appears spontaneously at around 40 days of age; this capacity can, however, be anticipated at 20 days of age if guppies are reared in an environment that offers such experiences from birth (Bisazza et al. 2010).

Though none has been specifically designed to test the hypotheses of Leibovich et al., several fish studies provide information relevant to the present debate. Concerning the existence of a holistic system for processing numerical and continuous magnitudes, data on fish research generally support this view. For example, Agrillo et al. (2011) found that mosquitofish routinely integrate numerical information and continuous magnitudes. Their performance was more accurate when both pieces of information were simultaneously available, compared with when only numerical information or only continuous information was provided. However, not all continuous magnitudes are equally important. Mosquitofish appear to rely on the total surface area and convex hull, whereas total luminance and contour length appear to be irrelevant. Interestingly, interindividual differences in the use of continuous magnitudes were observed in this species (Agrillo et al. 2009). Likewise, fish can rapidly discriminate four from five companions (Lucon-Xiccato et al. 2017), but their performance drops dramatically when prevented from using the total surface area or total activity of the stimulus fish (Agrillo et al. 2008).

Regarding the question of whether number sense is innate, available data on fish seem to contradict the authors' core hypothesis. In one experiment, 1-day-old fish were able to select the larger group of companions even when they were tested in an apparatus that allowed them to see only one fish at time, thus precluding the possibility of summing up areas or contours and gauging the density or convex hull of the groups (Bisazza et al. 2010). In another study in which guppies were trained in their first week of life to discriminate either between two numerical quantities controlled for the total surface area and other continuous magnitudes or between two figures differing in area by the same ratio (a condition in which numerical information was

made irrelevant), only fish trained with numbers learned how to discriminate (Piffer et al. 2013). Therefore, if a temporal mismatch between the number sense and the discrimination of continuous magnitudes does exist, in fish this appears to be opposite to that predicted by the model.

We acknowledge that the aforementioned data were not collected with these working hypotheses in mind and that alternative explanations are available in some cases. Nonetheless, we believe the cited examples convincingly demonstrate the possibility of investigating in fish the interesting issues raised by Leibovich et al. in a way that cannot as easily be done in higher vertebrates.

The number sense is neither last resort nor of primary import

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Abstract: Leibovich et al. argue that evidence for an innate sense of number in children and animals may instead reflect the processing of continuous magnitude properties. However, some comparative research highlights responding on the basis of numerosity when non-numerical confounds are controlled. Future comparative tests might evaluate how early experience with continuous magnitudes affects the development of a sense of number.

Leibovich et al. offered the provocative thesis that the idea of an innate "number sense" in humans and other animals may be misleading given the empirical tests that are used to assess such a sense. They proposed that most research in this area confounds numerical properties of stimuli with continuous properties such as area and density, and it is these continuous features that control responding more than true numerosity. We agree, in principle, with this position regarding typically used methods in comparative research, having also argued that tasks given to nonhuman animals may not be directly related to number, concepts but instead rely on non-numerical cues to guide responding (Beran et al. 2015a; 2015b). We also agree that comparative contributions to the broadly defined area of "numerical cognition" research need to carefully assess the competencies that are reported in adult humans, human children, and nonhuman animals by paying close attention to non-numerical confounds that may contribute to performance on these tasks (Beran & Parrish 2016).

However, we also believe that there are instances in which judgments by some animals are made on the basis of numerosity, where careful controls have eliminated the possibility they are using non-numerical, continuous quantitative information. For example, Beran (2012) showed that chimpanzees listened to food items being dropped into an opaque container and then compared that number of items with a visible, static set and chose the larger amount even though there were no continuous properties that would account for such performances (also see Beran et al. 2008). Animals also were trained with number symbols representing specific cardinal values, rather than specific magnitudes of stimuli. These include Arabic numeral-based studies with chimpanzees in which they labeled arrays of items with numerals (e.g., Matsuzawa 1985) and even combined multiple sets of items before labeling them (e.g., Boysen & Berntson 1989). A parrot also learned to vocally label arrays, even when queries about the number of items involved subsets of a specific class of items within a larger array (e.g., reporting the number of blue keys in a mixed array of blue and red keys and trucks [Pepperberg

1994]). In addition, chimpanzees learned to collect sets of items to match a presented Arabic numeral using a computerized enumeration task (e.g., Beran & Rumbaugh 2001). In these cases, we argue that numerosity is the dimension to which animals are responding, even though their performances typically showed the same "fuzziness" in representational acuity that is seen in magnitude judgment tasks (see Cantlon et al. 2009b), and training often took months or years to establish. These studies suggest that number concepts can emerge in a variety of nonhuman species, but may not do so as easily as representations of nonnumerical magnitudes.

When numerical and non-numerical properties co-vary in tasks, this does not mean that non-numerical properties must be controlling responding. Brannon and Terrace (2000) showed that across combinations of stimulus presentation formats where density, area, size, and contour length each co-varied to different degrees with number in a relative discrimination task, rhesus monkeys still used number in some cases. Beran (2007) showed that after training monkeys to discriminate which of two sequentially presented arrays of items on a computer screen was greater, the monkeys' performance withstood the introduction of trials where item size and rate of presentation offered potential distractions if monkeys were using those properties. This does not mean that monkeys or other animals do not use non-numerical properties when they can. And they may even preferentially use them, a point that is also true for human performance in many tasks. But when such performance can survive the control of such confounds, it suggests that a number sense is at work, even if it is not the dominant sensory signal to which a nonhuman animal may be attuned. The question then is whether an innate sense of number requires that numerosity operates as a dominant stimulus property in the face of other properties that compete to control responding. We do not see why this must be true.

Hence, we do not advocate abandoning the idea of an innate number sense in some nonhuman species or in humans. Rather, we acknowledge that this number sense likely emerged through evolution from a more generalized analog magnitude system for perceiving and representing quantitative information, and that the "number sense" and "magnitude sense" of nonhuman animals both suffer from the same processing and representational constraints (Cantlon et al. 2009b). In this perspective, an innate number sense can exist, and can be activated and used by many species, even if it is not a dominant conceptual system within a cognitive architecture. At the same time, this does not mean it is a "last resort" (Davis & Memmott 1982), but rather that it is an available system that can be accessed when other more generalized perceptual-discriminative processes for magnitude-based stimuli are insufficient to generate adaptive responses. For example, Cantlon and Brannon (2007) presented rhesus monkeys with a number matching-to-sample task in which monkeys could match on one of two dimensions, including number and some non-numerical cue such as color, surface area, or shape. Although monkeys preferred color and shape cues to number cues, they responded on the basis of number when the numerical difference between the sets was high (and therefore easier to discriminate).

The alternate view offered in the target article (and see Leibovich & Henik 2013) that a sense of number is not present in nonhuman animals (or perhaps even humans) at birth, but emerges through experiencing the correlation between numerosity and continuous magnitudes is an interesting proposal. If true, developmental research with pre-counting age children and a longitudinal comparison of true numerical processes for individuals with and without a history of experiences with continuous quantities would be particularly valuable. The latter approach would be a test that could only be conducted with nonhuman species, for obvious reasons, and would offer another chance for comparative research to make an important contribution.

Evidence for a number sense

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Abstract: Numerosity is inherently confounded by related stimulus attributes such as density and area, and many studies have reported interactions of various strengths between area, density, and numerosity. However, direct measurements of sensitivity within the area-density-numerosity space show that numerosity emerges as the most spontaneous and sensitive dimension, strongly supporting the existence of a dedicated *number sense.*

In the target article, Leibovich et al. rightly conclude that researchers should not simply assume the existence of a number sense, but put the hypothesis to the test. Cicchini et al. (2016) recently did so in a direct and rigorous manner. Borrowing from color science, they measured the equivalent of "MacAdam ellipses": discrimination thresholds in two-dimensional color space, whose short axis corresponds to the most sensitive direction, implying the action of specific mechanisms (MacAdam 1942).

Numerosity is the product of density and area, which defines a two-dimensional "numerosity space," with numerosity following the positive diagonal (Fig. 1). We measured discrimination thresholds within this space, using an "odd-one-out" technique (see Fig. 1A). Discriminations are well described by an elongated ellipse, with the short axis defining maximum sensitivity aligned to the numerosity diagonal. Sensitivity along this axis was 16 times higher than the orthogonal direction, showing that numerosity is the most sensitive dimension: just as red-green is a sensitive direction in color-space.

There was no a priori guarantee that numerosity would be the most sensitive dimension. This was clearly demonstrated by the fact that at high densities, where items were too crowded to be segregated, and numerosity discriminations are subject to different psychophysical laws (for review, see Anobile et al. 2016c), the results were quite different. The discrimination ellipses become more circular and are well predicted by independent encoding of area and density (Cicchini et al. 2016). The clear differences in processing of sparse and dense arrays highlight the action of specialized mechanisms for numerosity, which operate only under conditions where the items can be perceptually segregated.

Cross-modal and cross-format studies lend further support for the number sense, which may be more generalized than previously thought. Adapting to rapid or slow sequences of auditory or visual elements strongly affects the apparent numerosity of subsequent sequences of items, both visual and auditory (Arrighi et al. 2014). Importantly, the numerosity of spatial arrays is also affected by adaptation to temporal sequences, implying that the number of items – however presented – is encoded by a truly abstract system that transcends sensory modality, as well as space and time.

It is, however, important to note that the existence of specialized numerosity mechanisms does not preclude the possibility of interactions with other related attributes. Cicchini et al. (2016) also measured discrimination ellipses with a more subjective technique, where subjects were asked to make explicit judgments about numerosity, density, or area for stimuli within the area/density space. Clear interactions emerged. Discrimination boundaries for number judgments were not oriented exactly along the number axis, but at 37°, slightly toward the area axis and away from density (by about 17%), agreeing with other studies (e.g., Dakin et al. 2011; Gebuis & Reynvoet 2012c). However, the boundary for area judgments was drawn toward number by 53%, suggesting that number was as important as area in judging area, and that for density was drawn toward number by 78%, suggesting that density judgments are mediated by
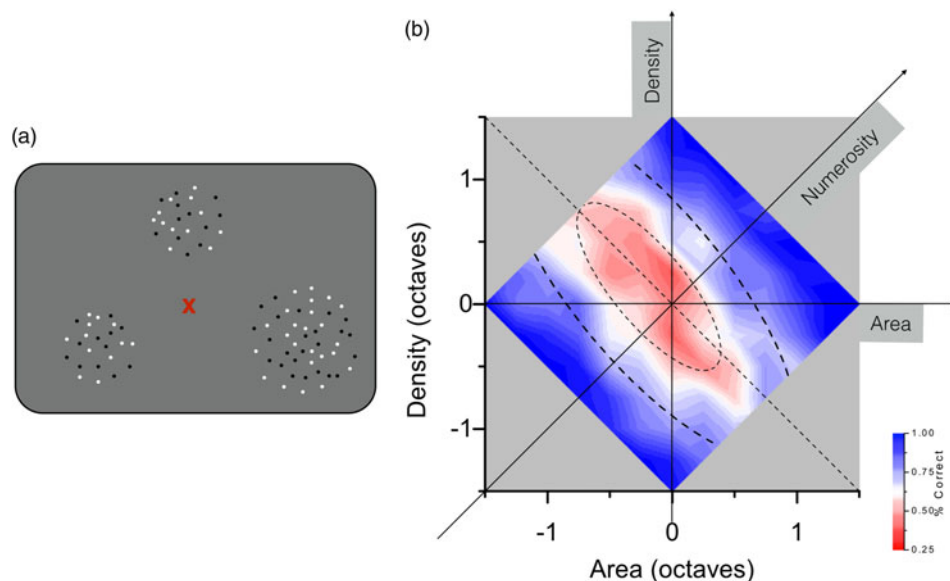
Figure 1. (Burr). Discrimination boundaries in two-dimensional log area-density space. (A) Example of stimuli used in the odd-one-out task. Two are identical "standards," defined as the origin of the space of panel B (24 dots, 40 square degrees); the other is the target, randomly positioned (lower right in this case), defined by a randomly chosen position in the colored part of the space (double numerosity in this example). (B) Thresholds in log area-density space. The abscissa defines the logarithmic difference in area between target and standard, and the ordinate, the logarithmic difference in density. Numerosity, the sum of log density and log area, increases along the positive diagonal. The heat map plots percent correct performance for all conditions. The two ellipses show 50% and 75% right responses, obtained from a two-dimensional Gaussian fit to the data. The aspect ratio of the ellipses is 4:1 octaves, implying a 16-fold difference between discriminations in numerosity and the orthogonal direction. Reproduced with permission from Cicchini et al. (2016).

numerosity (rather than the other way round). These results are not at odds with those discussed in the target article showing that number estimates can be slightly influenced by area or density, but indicate that interactions in the other direction are in fact far stronger: Number strongly influences both area and density, suggesting that number is the more basic attribute.

Interactions between seemingly unrelated perceptual attributes abound: Apparent speed depends on luminance, contrast, and color (Gegenfurtner & Hawken 1996; Thompson 1982); interval duration depends on speed (Brown, 1995; Kanai et al. 2006; Kaneko & Murakami 2009); event duration depends on size (Xuan et al. 2007); number depends not only on size and density, but also on eye movements (Binda et al. 2012; Burr et al. 2010) and the region of visual space where stimuli are displayed (Dehaene et al. 1993; Hubbard et al. 2005). More recently, we demonstrated interactions between action and number estimation, for both sequentially presented items and spatial arrays (Anobile et al. 2016a), pointing to even more far-reaching interconnections.

The fact that systems interact with each other does not preclude the existence of specialized mechanisms. That duration depends on speed and size does equate time with space, and very few would deny the existence of dedicated motion mechanisms, despite clear interactions with contrast, color, form, and time (Burr & Ross 1986; Burr & Thompson 2011). As Leibovich et al. point out, Harvey et al. (2013) have demonstrated a clear neuronal map for numerosity, robust to changes in low-level features. More recently, they have shown that the same area contains a tuning map for size (Harvey et al. 2015). The two maps do not superimpose and may not involve the same neurons; however, they coexist in the same area of brain, highly consistent with the notion of a brain area dedicated to magnitude estimation (Walsh 2003). The mapping for the two properties is clearly separable: This would allow for the independent estimation of size and number, but with a certain amount of cross talk.

To conclude, a great deal of evidence suggests that humans perceive number spontaneously, with dedicated mechanisms. Whether these mechanisms are innate is harder to prove.

However, developmental studies show that thresholds for numerosity discrimination are more adultlike at 6 years of age than are those for dense-texture discrimination (Anobile et al. 2016b), reinforcing other studies (Izard et al. 2009) reporting early emergence of number discrimination. Importantly, precision for numerosity, but not texture density, correlates with mathematics achievement in school-age children (Anobile et al. 2016b), adding weight to the idea that numerosity mechanisms act as a "start-up tool" for later acquisition of mathematics (Halberda et al. 2008; Piazza 2010). This strong link with mathematics provides a further motivation to understand fully the mechanisms underlying the perception of number and, possibly, the foundations for mathematics.

Approximate number sense theory or approximate theory of magnitude?

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Abstract: Leibovich et al. argue that the evidence in favor of a perceptual mechanism devoted to the extraction of numerosity from visual collections is unsatisfactory and propose to replace it with an unspecific mechanism capturing approximate magnitudes from continuous dimensions. We argue that their representation of the evidence is incomplete and that their theoretical proposal is too vague to be useful.

Leibovich et al. review evidence about the extraction of numerosity from visual sets and question the view that numerosity is apprehended directly, through a dedicated system, from the number of elements in the display. They propose instead that numerosity

results from the combination of several continuous dimensions that correlate with number, via a system that is built and refined throughout development.

Although we praise the enterprise, which makes current debates in the field visible, the empirical review does not do justice to the wealth and complexity of the evidence. As their declared aim is to challenge what they refer to as "number sense theory," Leibovich et al. often appear selective and biased in their overview. For example, they reject Nys and Content's (2012) claim that both number and area are extracted. In this study, subjects compared collections either for total area or for numerosity, and the congruency between number and aggregated area was manipulated orthogonally. The results showed effects of the irrelevant dimension in both tasks, suggesting that both dimensions are extracted automatically. Leibovich et al. discard this conclusion, citing as counterevidence studies that showed the influence of continuous dimensions on comparison performance, but did not examine the interference of numerosity on area. They further support their argument by referring to two studies that directly compared the interference of each dimension (number and area) on judgments of the other. Hurewitz et al. (2006) reported no numerical interference on area comparison, but in their stimuli, the largest numerical ratio was half as large as the largest area ratio. Leibovich et al. (2015) used numerosities in the subitizing range (2–4), so the largest numerosity ratio was 2:1. Based on the three examples of incongruent pairs in their figure 1, we calculated that the total area ratios were around 5:1. Both studies thus failed to adequately match the range of variation of the two dimensions, and our point still holds.

Leibovich et al. argue that various continuous dimensions such as cumulated area of the elements, length of the contour, and density, are most often confounded with numerosity, and that

they affect comparison performance despite efforts to control them. The conclusion is undisputable (and undisputed). But whether these effects have an impact on early extraction processes or later decision mechanisms is currently unclear. In adults, the influence of continuous cues seems much more limited in estimation than in comparison tasks (Gebuis & Reynvoet 2012c). Moreover, Content and Nys (2016) observed no influence of continuous cues at all in nonverbal numerosity estimation with 4-year-olds.

Therefore, the influence of continuous dimensions does not challenge the hypothesis that numerosity is automatically extracted and used, as Nys and Content's results suggest (see also Cicchini et al. [2016] for a similar argument). Leibovich et al. seem to have neglected some of the most convincing findings in this regard. Strong behavioral evidence that numerical magnitude is extracted independently of continuous dimensions comes from studies showing that numerosity estimation is changed by connecting some elements together without any modification, which would alter the continuous cues (Franconeri et al. 2009; He et al. 2009, Kirjakovski & Matsumoto 2016). Adaptation effects (Anobile et al. 2016c) also indicate the existence of a dedicated perceptual mechanism for numerosity.

Leibovich et al.'s conclusion is neither new nor controversial. It does not discredit number sense theory, as the authors acknowledge in the concluding section. Continuous dimensions are indeed most often correlated with number in our experience of the world. No wonder that we would tend to use them, when possible, in comparing collections. Rather, the thrust of the theory is that humans and other species are equipped with a mechanism making it possible to extract and encode discrete magnitudes *directly* from sensory stimulations. The proposal put forward by the authors aims at offering an alternative to number sense
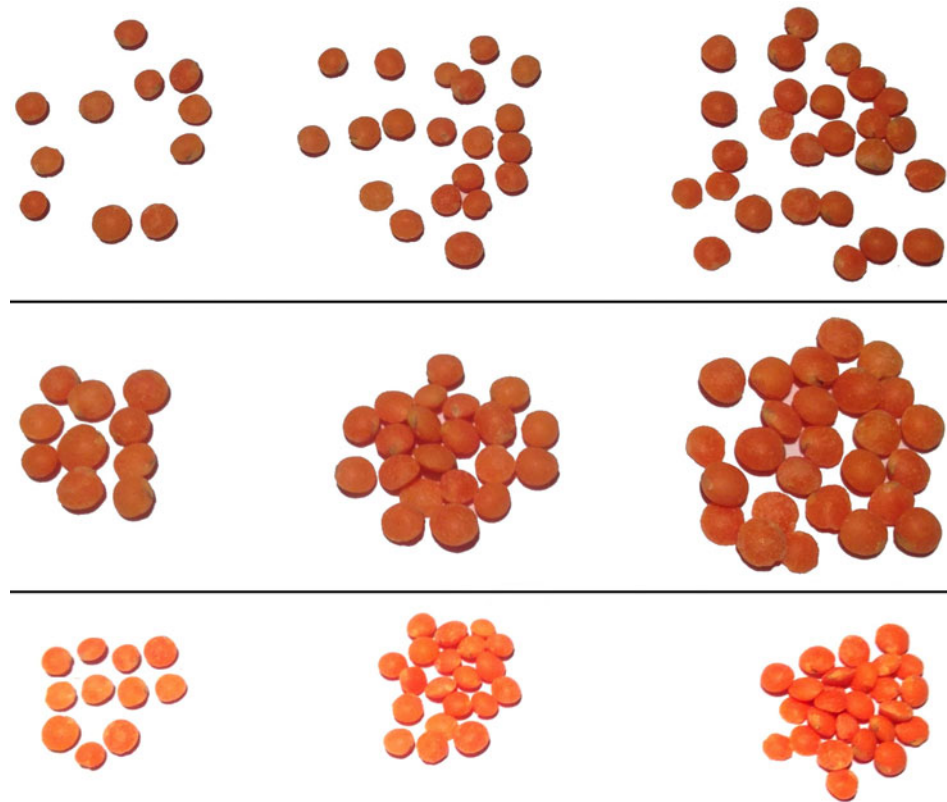

Figure 1. (Content et al.). Three examples of collections varying in numerosity. In each triplet, there are correlations between numerosity, contour length, occupancy, aggregated area, and so forth. These continuous cues can help order the collections, but they cannot serve to determine the numerosity, as the nature of the predictive relation changes from one triplet to another.

theory. It takes the form of a developmental scenario, whereby numerical competence emerges from learning "the correlation between numerosity and continuous magnitudes."

There are at least two logical flaws with this view. The first is its circularity. Capturing the correlation between numerosity and other dimensions requires representing numerosity in the first place. The authors acknowledge it and simply propose that number words would serve to trigger the emergence of numerosity. But how would number words make contact with numerosities? No answer is offered. The second and even more serious issue is that it is unclear whether continuous dimensions are sufficient to extract a representation of number, because the natural correlations between numerosity and continuous magnitudes even if most often present, are not stable in the world (see Figure 1). Sometimes numerosity could be predicted from contour area rather than occupancy, and sometimes it is the other way round. Therefore, whereas multiple cues may serve as proxies to order collections by numerosity, it is unclear how they could subserve the *estimation* of number.

To conclude, we would argue that a minimal requirement for future theoretical endeavours about numerosity processing would be to seriously consider and implement distinctions between the mechanism of numerosity extraction, the format of numerosity representation, and the decision processes that are required to perform a given task. That (some) continuous magnitudes would be extracted and combined in some weighted average to deliver a representation of number is one logical possibility. How the weights are determined without reference to numerosity remains, however, to be clarified. Yet another possibility would be that continuous magnitude information only affects late decision stages. Other scenarios are also possible, and we believe, more plausible. One is a specific, direct, numerosity extraction mechanism based on sampling the visual scene for individual elements feeding into a common magnitude representation system (see Cantlon et al. 2009b).

Perceiving numerosity from birth

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Abstract: Leibovich et al. opened up an important discussion on the nature and origins of numerosity perception. The authors rightly point out that non-numerical features of stimuli influence this ability. Despite these biases, there is evidence that from birth, humans perceive and represent numerosities, and not just non-numerical quantitative features such as item size, density, and convex hull.

Although it is impossible to simultaneously control for all continuous quantities in a single numerosity display, some studies have developed ingenious designs controlling these variables across all of the experiment's displays, as for example in Xu and Spelke's (2000) seminal study. Six-month-old infants saw first several arrays of a fixed numerosity (either 8 or 16, in different groups),

varying in dot size and position. Once habituated, all infants were tested with two numerosities in alternation (8 and 16). Crucially, different aspects of stimuli were controlled in the habituation and test phases: the summed area of all dots (as well as brightness and contour length) and the array area were matched on average between the 8 and 16 habituation groups, while the density and dot size were matched between the two tested numerosities. Therefore, if infants attend to dot size or density, they will respond in the same way to test numerosities 8 and 16; whereas if infants attend to summed area or array area, the two groups will respond similarly to the test stimuli. Sensitivity to non-numerical parameters, either a single parameter or a combination of them, thus cannot explain the interaction pattern observed: In both groups, infants looked longer at the novel numerosity. This finding has been replicated by a different group (Brannon et al. 2004), using different numerical values (Xu 2003), in the auditory modality (Lipton & Spelke 2003), and the same parameter control strategy was employed to demonstrate sensitivity to numerosity at the brain level (Izard et al. 2008, Piazza et al. 2004).

Similar controls for non-numerical features were used to demonstrate newborns' sensitivity to number (Izard et al. 2009). While hearing a fixed value of numerosity (e.g., 12), newborns looked longer to arrays matched in numerosity than to non-matching arrays (e.g., 4). Because the stimuli were presented across two different modalities (auditory and visual), the newborns' response was necessarily based on an abstract property of the stimuli. Following the logic of Xu and Spelke (2000), extensive parameters were controlled in the auditory stimuli across the two groups by equating the duration, and intensive parameters across the two test numerosities in the visual modality by equating density and item size. Therefore, infants' preference for the matching stimuli could be explained only by numerosity, not by sensitivity to an abstract notion of amount, or rate. Moreover, as infants received only one numerosity in the auditory modality, they could not be responding to relative quantity ("more" or "less"). In that respect, the numerosity paradigm departed crucially from another paradigm used later (de Hevia et al. 2014), in which newborns matched two values, one small and one large, across the two dimensions of numerosity and spatial extent. Newborns are able to relate increases versus decreases of quantities at a generic level, but also to perceive numerosities, calibrated across senses.

In line with these findings, studies investigating newborns' visual perception have demonstrated that they are able to represent individual objects, at the same age as in the numerosity study. In particular, human newborns can perceive complete shapes over partial occlusion (Valenza et al. 2006), and they can both distinguish individual elements of a stimulus or group them into a holistic percept (Antell and Caron 1985, Farroni et al. 2000, Turati et al. 2013). Moreover, newborns respond differently to faces displaying direct versus averted gaze (Guellai & Streri 2011), a much finer cue than the shapes used in the numerosity experiment. Perceptual abilities to individuate items from the background and from one another likely fed into the numerosity percept evidenced by Izard et al.'s (2009) study.

Despite the common belief that numerosity perception must be more complex, and therefore a later developmental achievement, than the perception of continuous quantity, developmental studies have provided evidence that numerosity discrimination is easier and more automatic. In particular, infants show higher sensitivity to, and prefer to look at, changes in numerosity over changes in item or total surface area, when difference ratios are equated across dimensions (Brannon et al. 2004; Cordes & Brannon 2008; 2011), and even when variations in number are smaller (Libertus et al. 2014). Similarly, children show higher sensitivity to number than to density (Anobile et al. 2016b). That perception of numerosity is more automatic than other continuous quantities is true in adults too: Even without an explicit task,

numerosity of visual arrays is processed faster than other continuous features of those arrays (Park et al. 2016b). In this context, it is important to note that although Stroop studies on adults indicate that continuous quantities interfere with number perception, much of the behavioral and neuroscientific evidence cited by Leibovich et al. is based on interference paradigms in which non-numerical quantities varied by considerably larger ratios (and, thus, likely had higher perceptual discriminability and salience) than numerosity.

At the brain level, areas in the intraparietal sulcus respond to numerosity, and not simply to non-numerical cues. In particular, Eger et al. (2009) used intraparietal sulcus activations to train a classifier to discriminate between patterns evoked by different numerosities across which item size was equated and found that this classifier generalized without accuracy loss to patterns evoked by numerosities across which total surface area was equated (and vice versa). Numerosity was also decodable from the intraparietal sulcus when low-level factors such as contrast energy were equated (Castaldi et al. 2016). Finally, in the right superior parietal lobe Harvey et al. (2013) observed an orderly topographical structure of numerosity responses, correlated across stimulus sets implementing different controls. Although the same region also responds to object size (Harvey et al. 2015), the tuning curves and map organization differ, thus highlighting the specificity of the numerosity response.

In summary, the literature brings uncontroversial evidence that humans perceive and represent numerosity from birth on. As pointed out by Leibovich et al., the literature also brings uncontroversial evidence that numerosity perception is imperfect, often subject to the influence of non-numerical aspects of stimuli. These phenomena are fascinating, as they open up a new research agenda – if perception of numerosity relies on an imperfect algorithm, we now need to crack up its functioning.

Multitudes are adaptable magnitudes in the estimation of number

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Abstract: Visual number comparison does not require participants to choose a unit, whereas units are fundamental to the definition of number. Studies using magnitude estimation rather than comparison show that number perception is compressed dramatically past about 20 units. Even estimates of 5–20 items are increasingly susceptible to effects of visual adaptation, suggesting a rather narrow range in which subitizing-like categorization processes blend into greater reliance on adaptable magnitude information.

When people perceive a collection as having an amount, do they assign a conceptual category (number) to something that is experienced as a multitude of units, or is that conceptualization dependent on language? In Book 7 of Euclid's *Elements* (300 BC/1956), Euclid famously defined a number as "a multitude of units" after having defined a unit, quite wonderfully, as "that by virtue of which each of the things that exist is called one" (p. 277). Leibovich et al. propose that whether nervous systems treat perceptual number as a multitude rather than a magnitude may be unknowable because perceived number cannot be isolated from all confounding perceptual continuous magnitudes that are typically correlated with number. But multiple information-processing systems in perception might work together to help obviate this concern. Here I consider how the fragile boundary between

magnitudes and multitudes might be manifest in numerosity estimation.

Unlike most perceptual magnitudes (loudness, area, brightness), numerosity has a built in unit. To compare the numbers of two collections is to try to identify a relative quantity of units. For small collections of two or three, special geometrical or attention processes may differentiate categories, but for large numbers, clearly any estimate must be an approximation. Is it simply a sensed magnitude? There is evidence that even a collection as small as five fails to form a discriminable numeric category in human adults in the absence of linguistic labels (Gordon 2004).

For some, the adaptability of visually perceived number is to strongly suggest that large visual number is estimated based on correlated features (Durgin 1995). How else could 200 dots appear perceptually equivalent to 400 dots? It could not be that some of the dots are missing. Rather, some visual property is clearly being adapted, and locally rescaled, and that property seems to act like a continuous magnitude (like brightness, loudness, etc.). Durgin argued that effects of adaptation produced multiple visual consequences including the underestimation of apparent numerosity – which was most pronounced for high numbers (in the hundreds), but also changes in perceived spacing or distribution. Adaptation, like number comparison, provides no obvious way to unconfound number, except insofar as adaptation fails (i.e., true number triumphs).

Number comparisons may be thought of as comparing several visual magnitudes correlated with numerosity (including area, Allik & Tuulmets [1991], and density). Whereas Anobile et al. (2014) sought to distinguish between number perception and density perception using differential Weber fractions, as Leibovich et al. point out, even distinguishing two distinct sources of judgment does not show that either one of them is number itself.

Still, the existence of multiple sources of information relevant to estimating numbers does not show that number perception does not occur. Having multiple sources of information about depth that get combined into a common perceptual estimate does not mean that we do not perceive depth, but it is hard to infer the information content of perceptual experience solely from discriminations tasks or categorization tasks.

An alternative approach to studying number with humans is to use magnitude estimation rather than magnitude discrimination. That is, human participants who have a linguistic number system can estimate how many units are present, just as they can estimate other psychophysical properties. Studies by Krueger (1972) and by Kaufman et al. (1949) have shown that dot collections as high as 200 dots are grossly underestimated, suggesting that "number" is (under) estimated rather than sensed for numbers of this magnitude. Perhaps this is just a translation problem of converting perceptions into words or maybe approximate "number" perception is just an adaptable continuous magnitude that humans conceptualize as being composed of units.

Alex Huk and I (Durgin 2016; Huk & Durgin 1996) tested how density adaptation affects number estimation. Participants who were adapted to dense texture to one side of fixation were briefly shown either one field of dots on one side or the other, or two fields of dots (one on each side). When only one field was flashed, they reported its apparent numerosity; when both fields flashed, they were to indicate which side appeared more numerous. The effect of adaptation on numerosity comparison was stronger as numerosity increased, and a similar pattern emerged for numerosity estimation.

The estimation data are shown in Figure 1. Number estimates were unaffected for 5 dots. But for more numerous collections (40 dots or more), estimates were about 25% lower in retinotopic regions adapted to dense (high numerosity) random dots fields than in unadapted regions. The average estimate for 256 actual dots, for example, was 154 in the unadapted region, and only 117 in the adapted region. Significantly, the numerosity estimation functions shown here in log-log space seem to bend significantly between 20 and 40 dots.
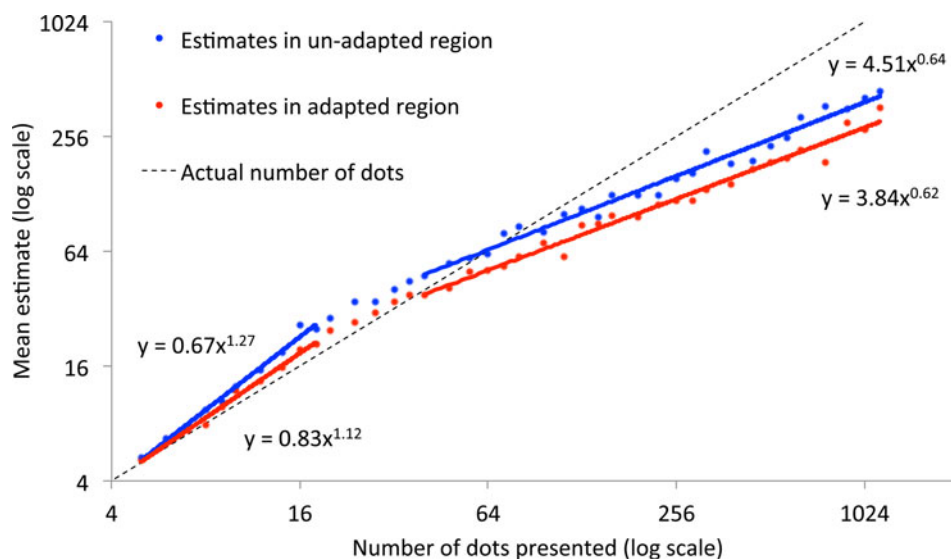
Figure 1.   (Durgin) Number estimation in adapted and unadapted regions (Durgin 2016; Huk & Durgin 1996). Data are re-plotted in a log-log scale and fit with power functions for values of 5–18 dots or 40–1,152 dots.

So what can we learn from magnitude estimates of numerosity? Magnitude estimation (i.e., assigning linguistic or symbolic numbers) for large non-symbolic numerosities behaves much like magnitude estimation data for other psychophysical magnitudes. It is consistently found that numeric estimates of large numbers of texture units are compressive in their scaling (have a slope less than 1 in log-log space). Additionally, the break between low and high numbers depicted in the graph is quite dramatic, and it encourages us to think, as Anobile et al. (2014) also seem to propose, that numerosity is not a single perceptual dimension. In visual number perception, at least, number investigation probably ought to think of numbers higher than about 20 as perceptual magnitudes, not multitudes. But what does the growing effect of adaptation mean between 5 and 20? Five uniformly colored dots seem to be unaffected by adaptation. Our subjects, who have a linguistic number system, found this multitude of units easy to identify even when briefly flashed peripherally in a dense-adapted region. Beyond 5, performance seems quite different. Beyond 20, it seems different again.

NOTE
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Why try saving the ANS? An alternative proposal

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Abstract: Leibovich et al. propose that continuous magnitudes and a number sense are used holistically to judge numerosity. We point out that their proposal is incomplete and implausible: incomplete, as it does not explain how continuous magnitudes are calculated; implausible, as it cannot explain performance in estimation tasks. We propose that we do not possess a number sense. Instead, we assume that numerosity judgments are accomplished by weighing the different continuous magnitudes constituting numerosity.

How do we approximate large numerosities? The dominant view is that this is accomplished via the "approximate number system" (ANS), an innate number system that is able to process pure numerosity (e.g., Dehaene 2003). Recently, others and we highlighted the role of continuous magnitudes (e.g., density, size, total surface of the dots, etc.) in numerosity judgments (e.g., Allik & Tuulmets 1991; Gebuis et al. 2016; Gevers et al. 2016). Leibovich et al. therefore challenge, in the current article, the idea that number sense is innate. They note that a natural correlation exists between continuous magnitudes and numerosity and argue that both types of information are used "holistically"[1] to judge numerosity. Their ideas are presented in a developmental framework within which processing of continuous magnitudes is innate but a number sense develops over time.

We fully agree with the authors that something is wrong with the ANS theory. We also agree that continuous magnitudes play an important role. The proposed idea of holistic processing of numerosity using both a number sense and continuous magnitudes is appealing at first sight. However, a closer inspection of the proposal makes us conclude it is both incomplete and implausible.

First, the account is incomplete, as it does not explain exactly how continuous magnitudes are judged and how they would bias the numerosity estimate. One possibility is that subjects can estimate *all* continuous magnitudes (size, density, diameter, contour length, etc.) simultaneously. This would allow the subjects to make an exact calculation of the numerosity, and hence, the numerosity estimate should be free from any bias. This result is clearly inconsistent with the literature, and this possibility can therefore be rejected. Another possibility is that the subject only decides which of the two stimuli contains "larger" or "more" continuous magnitudes. This could indeed cause the observed bias, but how would this work in cases in which only a few continuous magnitudes are larger in one stimulus compared with the other? Take for example two stimuli with the convex hull being larger but the surface smaller in one stimulus compared to the other.

Do the subjects now rely only on the most prominent continuous magnitude? This is unlikely, as results have shown that the bias increases with the number of continuous magnitudes being manipulated (Gebuis & Reynvoet 2012b). Another alternative is that each continuous magnitude contributes to the final response relative to its size. Unless the authors have another suggestion on how continuous magnitudes could help in making numerosity judgments, the previously discussed reasoning leaves us with the latter solution. We already proposed this solution and implemented it in a model that relies only on continuous magnitudes (Gebuis et al. 2016; Gevers et al. 2016).

Second, the model provides an explanation only for comparisons tasks, but not for estimations (e.g., How many objects are presented?). We argue that performance in estimation tasks uncovers why the model is implausible. The model cannot explain how the biases or congruency effects as observed in a comparison task occur in estimation tasks. The reason is that in an estimation task, only a single numerosity is presented and the continuous magnitudes of this numerosity are not informative about the number presented. For example, knowledge about the diameter of the individual objects does not provide information about the numerosity presented. Related to the previous discussion, more detail is needed about how the continuous magnitudes influence a numerosity judgement and, more specifically, numerosity estimation. As outlined previously, an exact calculation/estimation of the exact size of each sensory cue would enable us to calculate the exact value presented. This means that our number sense and our continuous magnitudes would derive the same result and thus would not induce a bias. Furthermore, when a single set of objects has to be estimated, it is impossible to make a small/large judgment given that there is no other stimulus with which to compare it. The continuous magnitudes on their own are simply not informative for estimation and therefore should, according to the current model, not bias numerosity estimation. However, this is not right, as multiple studies have shown a bias induced by the continuous magnitudes when estimates are performed (e.g., Gebuis & Reynvoet 2012c; Izard & Dehaene 2008). The authors could argue that subjects calculate a running average based on the set of stimuli used in the experiment and compare the continuous magnitudes with this running average. However, even in a study where only a single stimulus was presented (and hence no running average could be calculated), a bias was induced by the continuous magnitudes (Krueger 1982).

The proposed review makes it clear that the influence of continuous magnitudes on numerosity processing challenges the ANS theory. However, the alternative proposal made by Leibovich et al. can be rejected based on both logical and empirical grounds. We therefore propose a different solution, which is more parsimonious, stepping out of the comfort zone where researchers try to adapt the idea of an ANS to preserve it. We put forward the simple suggestion that we do not possess nor develop an ANS. Instead, we argue that it is much more straightforward to assume that numerosity estimations or comparisons are accomplished by weighing the different sensory cues constituting numerosity, whereas language is used to describe this numerosity (for an extensive review on this matter, see Gebuis et al. [2016] and Gevers et al. [2016]). Contrary to the current model, it does explain how numerosity can be derived from the sensory cues. This proposal will therefore spark the debate on the ANS, not when it comes into existence, but if it exists at all.

NOTE
1. Note that Gebuis and Reynvoet (2012b), Gevers et al. (2016), and Mix et al. (2002a) are cited incorrectly. They do not support the claim that a number sense and sensory cues are processed holistically. Instead, Gebuis and colleagues suggest that numerosity judgments are based solely on continuous magnitudes, Mix et al. propose that number sense is not innate, and several ideas are proposed about how number sense could develop.

The evolvement of discrete representations from continuous stimulus properties: A possible overarching principle of cognition

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Abstract: Leibovich et al. propose that non-symbolic numerosity abilities develop from the processing of more basic, continuous magnitudes such as size, area, and density. Here I review similar arguments arising in the visual perception field and further propose that the evolvement of discrete representations from continuous stimulus properties may be a fundamental characteristic of cognitive development.

Ever since early research on the building blocks of visual perception, basic stimulus properties such as line orientation and movement have been shown to be mapped in a graded fashion within early visual cortex (e.g., Hubel & Wiesel 1977). Relatedly, a hallmark of spatial coding within this region is retinotopic mapping, in which the external space is represented continuously within neural tissue. With the emergence of hemodynamic neuroimaging techniques, investigation of the nature of higher-level object recognition mechanisms was accelerated, revealing several subregions within ventral temporal cortex that are dedicated to processing discrete visual categories, such as faces (Kanwisher et al. 1997; McCarthy et al. 1997), places/scenes (Aguirre et al. 1998; Epstein & Kanwisher 1998; Maguire et al. 1998), human body parts (e.g., Downing et al. 2001), and even letters/words (Fiez and Petersen 1998; Puce et al. 1996). These regions, or networks of functionally specialized processing units, are typically referred to as *domain-specific*, as they are uniquely (albeit not exclusively) activated by specific stimulus categories (e.g., Kanwisher 2010; Yovel & Kanwisher 2004) and are tied to rather well-defined anatomical structures (see Grill-Spector & Weiner 2014). The discovery of such functionally dedicated "islands," or modules, within ventral visual cortex served as a window to the understanding of the underlying mechanisms of high-level visual perception, as clearly, we perceive the world as a collection of meaningful, individuated objects grounded within background sceneries, not of arbitrary orientations, colors, or motion directions. Visual categorization, or the ability to detect and recognize objects by representing them in a separable, discrete fashion, is thus one of the key functions of the visual system (Grill-Spector & Weiner 2014; Kanwisher 2010).

Yet more recent advances have shifted the pendulum back to acknowledging the importance of domain-general processes in visual recognition, by revealing various large-scale mapping principles that may in fact underlie categorical representation along the ventral visual cortex. Thus, for example, one such organizing principle is visual field eccentricity, according to which face- and word-selective regions reside within foveally biased regions (because of their requirement for high-resolution vision), whereas place- and scene-selective regions lie along peripherally biased areas within the visual cortex (because of their dependence on large-scale visual integration [e.g., Hasson et al. 2002; Levy et al. 2001]). Other organizing principles are real-world object size (e.g., Konkle & Oliva 2012), motion/mobility (Chao et al. 1999; Huth et al. 2012), animacy (Connolly et al. 2012; Kriegeskorte et al. 2008), and semantic meaning (Huth et al. 2012). Importantly, a common feature of most of these organizing principles is their gradient nature, where categories are represented as locations in a continuous space mapped smoothly across the cortical surface. From a developmental perspective, it may well be the case that the domain-specific "modules" specialized to process faces or places have evolved from earlier gradient maps,

which encode more basic visual properties such as eccentricity and motion.

Indeed, the extent to which the ability to process "unique" categories is innate, or acquired with experience and with cortical maturation, is still heavily disputed. From an evolutionary perspective, the existence of certain hardwired modules, dedicated to processing a specific type of content, can be easily justified for at least some of the aforementioned domains (e.g., face processing is necessary for infant bonding with caregiver, body-part processing is important to action understanding and social interaction). When discussing face perception, which is at the heart of scientific controversy, studies have demonstrated a genetic basis for face recognition abilities (Shakeshaft & Plomin 2015; Wilmer et al. 2010; Zhu et al. 2010), along with prioritized processing of face-like configurations during early infancy (Goren et al. 1975; Johnson et al. 1991). Both types of findings strongly support an innate account of face processing. Other developmental research, however, has revealed a rather complex, protracted profile of face-expertise development, according to which face recognition behavior and its underlying neural mechanisms develop gradually during childhood and do not reach full maturation until early adulthood (e.g., Cohen Kadosh et al. 2010; Gathers et al. 2004; Germine et al. 2011; O'Hearn et al. 2010; Scherf et al. 2007). Furthermore, according to some models, despite their seemingly dissociated nature, face- and word-processing abilities emerge from shared, overlapping neural systems, both relying on high-acuity cortical regions (e.g., Hasson et al. 2002). It is only with formal education and the acquirement of orthographic symbols, which are represented most dominantly in the left hemisphere, that face-recognition mechanisms are "forced" to become right-lateralized and, consequently, attain partial independence (Behrmann & Plaut 2013; 2014; Dundas et al. 2013; 2014). According to this view, the cortical mechanisms that support visual cognition at adulthood develop from a common, distributed neural network, rather than from a set of distinct, innate regions that each subserve a particular visual function.

As for the discussion regarding the innateness and independence of a potential "number sense" mechanism, although there are clearly differences in the nature of numerosity and visual categorization abilities, a possible common feature of the two types of processing is their emergence from more basic, continuous representations, which gradually develop into discrete representational entities. As Leibovitch et al. propose in their new theoretical framework, developments such as parsing stimuli one from another (visual individuation), the acquisition of language (counting), and the ability to abstract an exact number of objects from their variant physical properties (abstraction and inhibition), eventually allow one to represent magnitudes in a discrete, numerical fashion. Clearly, much research is still required to fully characterize the developmental profile of continuous and discrete magnitude processing and their interdependent nature (see, e.g., Gabay et al. 2013; 2016; Pinel et al. 2004; Walsh 2003), along with their interactions with other domain-general systems (such as cognitive control mechanisms [e.g., Leroux et al. 2009]). The current model may serve as an overarching framework for such future study.

Magnitude rather than number: More evidence needed

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Abstract: Leibovich et al. do not present enough empirical support to overturn decades of work supporting a *number sense* nor to convince

the reader that a *magnitude sense* provides a better explanation of the literature. Here we highlight what we feel are the main points of weakness and the types of evidence that could be provided to be more convincing.

Leibovich et al. propose that findings of a *number sense* in human infants, children, adults, and nonhuman primates can be alternatively explained by a *magnitude sense*.

Possible or unclear is not enough. The main argument the authors lay out is that it is impossible to control for all non-numerical magnitude properties that co-vary with numerosity at once, and, as such, all previous work attempting to control for non-numerical magnitude properties to study numerical processing cannot be straightforwardly interpreted as having to do with "number." Unfortunately, simply saying that all magnitude dimensions were not controlled in a given study does not provide any empirical support for the position that the observed responses were non-numerical. To be taken seriously, the authors need to say not only which particular magnitudes confounded number in each example, but also why the empirical evidence favors that particular non-numerical magnitude interpretation over a numerical one in each of those cases. More broadly, the authors need to explain how alternatively explaining the literature by various and different non-numerical confounds in each experimental context provides a more comprehensive and consistent explanation of the literature than the number sense.

The authors also seem to misunderstand the controls used in neural adaptation (e.g., Piazza et al. 2004), where the impossibility of controlling for all non-numerical magnitude properties in a single stimulus (or pair of stimuli) is avoided by requiring a more complex pattern of responses across many stimuli (see Kourtzi & Grill-Spector 2005). In adaptation experiments, participants see a sequence of novel images that vary on many non-numerical magnitude dimensions. A majority are "adaptation" images with the same number of items. Occasionally, "deviant" test images are presented that deviate on numerical or non-numerical parameters (e.g., shape). Critically, adaptation and test images are drawn from different control distributions where number and non-numerical magnitude properties share different relationships to one another. Although researchers focus on equating different magnitude dimensions across adaptation and test image sets (e.g., item size and item spacing vs. total area and luminance), many magnitude dimensions are interrelated, and systematically changing relationships between some dimensions necessarily changes the others. For example, changing the relationship between item size and number also changes the relationship between number and item spacing, item contour length, total area, total contour length, brightness, and so forth. Even if a particular undefined non-numerical magnitude dimension was inadvertently confounded with number and led to attenuation in the brain response during the adaptation images, the relationship between that particular dimension and number would change in test images. Thus, one would not observe differential release from adaptation between numerical and non-numerical deviants (e.g., shape) because the relationship between the particular (undefined) non-numerical magnitude and number that was confounded during adaptation necessarily changes in all test types, as test images are drawn from a different set of controls. In contrast, if the brain response attenuates to repeated presentation of the same number of items *and* selectively recovers with a change in number despite these sorts of controls, it can be concluded that the response was numerical (see Kourtzi & Grill-Spector 2005).

An influence on number does not equal non-numerical. The main positive empirical evidence drawn upon by the authors to support the hypothesis that previously observed responses are non-numerical comes from studies showing *influences* of non-numerical properties on numerical judgments or neural processing (e.g., Gebuis & Reynvoet 2014; Leibovich & Henik 2014; Leibovich et al. 2015). Although this point highlights the fact that the

field currently lacks a model of how exactly number is extracted from visual arrays and provides evidence that non-numerical properties may be used to do so, evidence of an influence of non-numerical magnitudes on number is not evidence that the response is not numerical in nature. In a similar vein, the existence of a brain region that processes both numerical and non-numerical magnitudes (e.g., Walsh 2003) is not evidence that number is not represented distinctly at other levels or regions. To be taken seriously, the authors would need to provide evidence that responses thought to be numerical are not only influenced, but also better explained by non-numerical magnitudes.

No actual developmental evidence provided. The authors fail to provide any positive empirical evidence in support of their main developmental claim: Continuous magnitude representation is innate and number representation is not. Moreover, the authors disregard some of the strongest evidence to date for an innate number sense, cross-modal numerical processing in infants (e.g., Feigenson 2011; Izard et al. 2009; Jordan & Brannon 2006). Cross-modal number studies with infants are the smoking gun here, as they provide evidence of an innate number sense while avoiding the challenge of controlling for non-numerical magnitudes within a single sense modality. Leibovich et al. assert that cross-modal number studies with neonates (e.g., Izard et al. 2009) cannot be trusted because neonatal vision is too "blurry," not allowing object individuation until about 5 months. Furthermore, they claim that results from cross-modal numerical matching studies with infants are highly inconsistent and, therefore, should not be trusted. The claim that infants cannot perceptually individuate objects until 5 months is simply false. Moreover, the authors inaccurately estimate study reliability (not considering the different predictions of preferential matching vs. violation of expectation paradigms). In both cases, the cited references (Cantrell & Smith 2013; Carey 2001) do not support the authors' claims. The authors are right that neonatal vision is much worse than child or adult vision. However, to convince the reader to disregard the cross-modal number studies with neonates, the authors would need to provide actual evidence that typical newborn visual acuity is below the threshold needed to see the stimuli used in these particular studies (e.g., estimates from Fantz et al. [1962] and Kellman & Arterberry [2007] suggest otherwise).

In the words of David Hume (1748/2007), "A wise man [sic], therefore, proportions his belief to the evidence" (p. 80). Although the magnitude sense proposal is intriguing and highlights important gaps of knowledge in the field, the authors would need to present stronger empirical evidence to actually overturn the number sense literature and garner widespread support for their own view.

Is the ANS linked to mathematics performance?

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Abstract: Leibovich et al. argue persuasively that researchers should not assume that approximate number system (ANS) tasks harness an innate

sense of number. However, some studies have reported a causal link between ANS tasks and mathematics performance, implicating the ANS in the development of numerical skills. Here we report a *p*-curve analysis, which indicates that these experimental studies do not contain evidential value.

As Leibovich et al. point out, the dominant view is that mechanisms involved in performing "number sense" or approximate number system (ANS) tasks underlie the basis of symbolic mathematical skill. This view is based on findings from two main experimental paradigms: In comparison tasks, participants select which of two arrays contains more dots; in addition tasks, they assess whether two sequentially displayed arrays contain more dots than a third array. Researchers have assumed that such ANS tasks harness an innate sense of number, but Leibovich et al. argue that this assumption is not warranted.

In our view, there are three main sources of evidence for the view critiqued by Leibovich et al.:

1. Face validity. Tasks in which children or adults compare, for example, the number of yellow and blue dots do, on the face of it, seem to be about number.

2. Correlational evidence. Recent meta-analyses have reported that performance on standardized mathematics tests and ANS tasks correlate at $r = 0.2$ to 0.3 (Chen & Li 2014; Fazio et al. 2014; Schneider et al., 2017).

3. Causal evidence. Some recent experimental studies have claimed that improving performance on ANS tasks causes higher mathematics achievement and faster mathematics performance.

Leibovich et al. argue compellingly that evidence from comparison tasks is insufficient to conclude that ANS tasks involve numerical processing, at least not as currently conceived by proponents of the "number sense" theory (cf. Gebuis et al. 2016). Further, accounting for the ANS/mathematics achievement correlation does not require the assumption of an innate sense of number, because of the inhibitory control demands of incongruent trials on ANS tasks (e.g., Fuhs & McNeil 2013; Gilmore et al. 2013). However, this inhibition confound does not account for the third source of evidence, which Leibovich et al. do not address.

The causal evidence comes from two sources. One line of research has found training on ANS tasks leads to improved performance and faster responses on mathematics tests (Hyde et al. 2014; Khanum et al. 2016; Park & Brannon, 2013; 2014; Park et al. 2016a). Another has found that manipulating the order in which ANS trials are presented (easy-to-difficult or difficult-to-easy) improves mathematics performance (Wang et al. 2016). These findings present a problem for the inhibition account. Earlier research found that inhibition training does not transfer to non-trained tasks (Thorell et al. 2009), so potential inhibitory control demands of ANS tasks cannot explain these findings.

Although there has been a debate about the quality of this evidence (e.g., Lindskog & Winman 2016; Merkley et al. 2017; Park & Brannon 2016; Wang et al. 2017), here we ask whether, taken together at face value, current experimental studies provide sufficient evidence to conclude that there is a causal link between the ANS and mathematics performance. To this end, we performed a *p*-curve analysis on the set of all studies we are aware of that report a causal link between the ANS and mathematics performance (Hyde et al. 2014; Khanum et al. 2016; Park & Brannon 2013; 2014; Park et al. 2016a; Wang et al. 2016).

P-Curve analyses (Simonsohn et al. 2014; 2015) rely on the fact that *p*-values follow a uniform distribution under the null hypothesis. In contrast, when the null is false, *p*-values are right skewed (i.e., there are more low values than high values). This is true not only for the full 0-to-1 interval, but also for the interval from 0 to 0.05. Simonsohn et al. proposed that the shape of the distribution of significant *p*-values in a set of studies can be used to assess if they
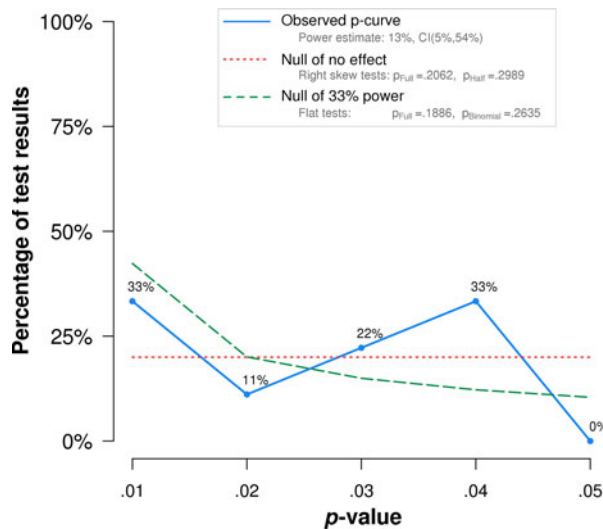
Figure 1. (Inglis et al.). Distribution of *p*-values for studies finding a causal connection between the ANS and mathematics performance. The observed p-curve includes nine statistically significant ($p < .05$) results, five of which are $p < .025$, No nonsignificant values were entered.

collectively contain evidential value. If the significant *p*-values follow a roughly uniform distribution, publication bias might explain the results.

We followed Simonsohn et al.'s (2015) procedure and, for each reported study, extracted the test statistic associated with the hypothesis of interest (whether the experimental manipulation influenced mathematics performance). If there was doubt about which statistic to select (e.g., the study contained two control groups), we conservatively selected the comparison with the smaller *p*-value (retaining the other for a robustness check). Details are given in our *p*-curve disclosure table at https://figshare.com/s/e6188771f048ecc6f881.

We analyzed the test statistics using the *p*-curve app v4.05 (http://www.p-curve.com/app4/). The *p*-value distribution is shown in Figure 1. Of nine *p*-values, five were below 0.025, a frequency not significantly different to the 4.5 expected under the null hypothesis, one-tailed binomial test, $p = 0.5$. Stouffer's method (Simonsohn et al. 2015) also indicated that these studies do not contain evidential value (*p*-values = 0.206, 0.299). The *p*-curve method also provides an estimate of the power of the studies. Here this was 13% (90% confidence interval: 5%, 54%), indicating insufficient evidence to reject the null of 33% power (which Simonsohn et al. would take to indicate that evidential value was absent and that replications would not be expected to succeed).

Our findings indicate that the published literature to date does not contain evidence of a causal link between performance on ANS tasks and standardized mathematics tests. To be clear, we have not demonstrated there is no causal connection between the ANS and mathematics performance, only that the existing literature does not provide evidence for one. However, we can definitively conclude that existing studies are substantially underpowered, rendering their interpretation difficult. In the future, researchers should address this limitation through preregistration and larger samples.

To conclude, we endorse Leibovich et al.'s suggestion that the assumption that ANS tasks involve number sense is not justified. Although Leibovich et al. did not address it, we believe that existing evidence of a causal link between the ANS and mathematics performance is insufficient to challenge their argument.

Magnitude, numerosity, and development of number: Implications for mathematics disabilities

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Abstract: Leibovich et al. challenge the prevailing view that non-symbolic number sense (e.g., sensing number the same way one might sense color) is innate, that detection of numerosity is distinct from detection of continuous magnitude. In the present commentary, the authors' viewpoint is discussed in light of the integrative theory of numerical development along with implications for understanding mathematics disabilities.

In their article, "From 'Sense of Number' to 'Sense of Magnitude': The Role of Continuous Magnitudes in Numerical Cognition," Leibovich et al. challenge the prevailing view that non-symbolic number sense (e.g., sensing "sixness" in a group of six objects in the same way one might sense color [Dehaene 1997]) is innate, that detection of numerosity is distinct from detection of continuous magnitude. Rather, the researchers argue that infants are hardwired to sense continuous magnitudes (e.g., objects' areas, contour lengths, spacing) and that this information supports perception of numerosity. Non-symbolic number sense "develops from understanding the correlation between numerosity and continuous magnitudes" (sect. 8, para. 6). For example, infants might learn through their everyday experiences that more objects typically take up more space. However, in cases where this rule is violated (e.g., a line of three long toy trucks that is longer than a line of five short trucks), general cognitive control or executive function may be needed to inhibit a response based on the expected correlation of continuous magnitude and numerosity. In the present commentary, we discuss Liebovich et al.'s viewpoint in light of the integrative theory of numerical development (Siegler & Lortie-Forgues 2014), which was not considered in the article, along with implications for understanding mathematics disabilities.

The integrative theory "proposes that the continuing growth of understanding magnitudes provides a unifying theme for numerical development" (Siegler & Lortie-Forgues 2014, p. 144). A mental number line, Siegler and Lortie-Forgues contend, coordinates knowledge of different forms of magnitude ranging from non-symbolic continuous quantities and numerosities to symbolic whole numbers, fractions, and decimals. Arguably, a mental number line could be grounded in core knowledge of continuous magnitudes, which allows children to perceive non-symbolic numerosities and eventually symbolic numbers. However, we believe that the argument that an innate sense of non-symbolic magnitude is more fundamental than non-symbolic number sense is less important than is the notion that an understanding of how discrete and continuous quantities are related is critical for constructing the beginnings of a mental number line. According to the integrative theory, the construction of a mental number line structures mathematics learning, and early experiences with both continuous magnitudes and discrete objects in the external world shape children's understandings of quantity right from the start.

Leibovich et al. present implications of their view for understanding mathematics disabilities, including dyscalculia, as well as mathematics disabilities that co-occur with other conditions (e.g., dyslexia or attention deficits). The possibility that children at risk for mathematics disabilities have trouble grasping the correlation between continuous magnitudes and numerosities is intriguing. Previous work has shown that core deficits in understanding numerical magnitudes in symbolic contexts underpin mathematics disabilities (e.g., Butterworth 1999; 2005; Butterworth & Reigosa-Crespo 2007; Landerl et al. 2004). For example, children with

dyscalculia perform much more poorly than do their typically achieving peers when asked to identify which of two numerals is larger and to relate quantities to their symbols (Butterworth & Reigosa-Crespo 2007; Landerl et al. 2004; Rousselle & Noel 2007). Moreover, ability to estimate the placement of numbers on a number line strongly predicts whether children will go on to struggle in mathematics (Hansen et al. 2015; Resnick et al. 2016). It is possible children's difficulties with symbolic representations of quantity may be at least partly rooted in the detection of non-symbolic continuous magnitudes. For example, Matthews et al. (2016) showed that mathematics competencies that depend on an understanding of fractions are predicted by non-symbolic processing of ratio information, which is inherently continuous.

Nonetheless, we generally argue against the "domain-general account of dyscalculia" discussed by Leibovich et al. Although general cognitive competencies partially explain why some children struggle with mathematics (Jordan et al. 2013; Rousselle & Noel 2007), basic weaknesses in understanding numerical magnitudes have been shown to be the more definitive characteristic of mathematics disabilities (Clarke & Shinn 2004; Hansen et al. 2015; Jordan et al. 2013; Mazzocco & Thompson 2005). We recognize, however, that children's core difficulties with numerical understanding are exacerbated by weaknesses in executive functioning related to inhibition and set shifting, which constrain children's later numerical development (e.g., Hassinger-Das et al. 2014). Being able to focus on number while ignoring irrelevant information helps children master foundational number skills more quickly. Preschoolers' "spontaneous focusing on numerosity" (SFON) predicts rational number understanding 6 years later (McMullen et al., 2015). Cognitive control may be especially important for learning fractions, where a larger number in a fraction's denominator may not always correspond to its magnitude (e.g., 2/6 is smaller than 2/3). In fact, attention emerges as a strong and unique predictor of fraction learning among a constellation of variables (Hansen et al. 2015; Jordan et al. 2013; Rinne et al. 2017). Not surprisingly, attention deficits frequently co-occur with mathematics disabilities (Zentall et al. 1994). Recognizing the correlation between continuous magnitudes and numerosities may be an important cue for determining numerical magnitude, but it is likely just one cue among a number of others, and an inability to use such cues in general likely reflects a domain-specific core deficit in processing magnitudes numerically.

Finally, the term *number sense* has been used by some to indicate a fundamental ability that has been interpreted as being relatively impervious to change. A similar interpretation might be made for "magnitude sense." However, we and others define number sense more broadly to include core symbolic knowledge of number and numerical magnitudes or relations (Jordan & Dyson 2016; National Research Council 2009). A growing body of experimental evidence – at least at the symbolic level, which is most closely related to learning mathematics – indicates that number sense can be developed in all or most children (Frye et al. 2013; Jordan & Dyson 2016). Rather than separating continuous magnitude sense from number sense, as Leibovich et al. propose, it might be more useful to view these understandings as interacting along a continuum of numerical development.

Infants discriminate number: Evidence against the prerequisite of visual object individuation and the primacy of continuous magnitude

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Abstract: Leibovich et al. hypothesize that the absence of visual object individuation limits infants' numerical skills and necessitates a reliance on continuous magnitudes. We argue that parallels between infants' numerical discrimination in the visual and auditory modalities, their abilities to match numerosities across modalities, and their greater ability to discriminate changes in number compared with continuous magnitudes contradict the authors' assumptions.

The authors present a provocative theoretical model describing how the development of the number concept unfolds based on children's experiences of the correlations between continuous magnitudes and number in their environment. This theory rests on the assumption that infants cannot readily represent number because they cannot individuate visually presented objects early in life and that continuous magnitudes do not suffer from this constraint and are hence easier to represent. We question this assumption based on three lines of empirical evidence: (1) parallels between infants' abilities to represent numerical information presented in the visual and auditory modalities, (2) infants' ability to match numerical information across sensory modalities, and (3) infants' greater facility in discriminating number compared with continuous magnitudes in the visual modality.

First, audition develops already in utero (Hepper & Shahidullah 1994), and infants are born with sophisticated auditory abilities such as the ability to discriminate their mother's voice from a female stranger's voice (DeCasper & Fifer 1980). With respect to numerical skills, 6-month-old infants are able to discriminate sequences of tones when the numbers differ by a 1:2 ratio (e.g., 8 tones vs. 16 tones) but fail when they differ only by a 2:3 ratio (Lipton & Spelke 2003). Auditory numerical stimuli obviate the need for individuation because tones are typically presented sequentially. However, infants' ability to discriminate auditory numerical stimuli is remarkably similar to their ability to discriminate simultaneously presented visual arrays of objects (i.e., 6-month-old infants can discriminate visually presented numerosities that differ by a 1:2 ratio, but not a 2:3 ratio [Xu & Spelke 2000]). If, as hypothesized by Leibovich et al., visual object individuation only emerges around 5 months of age and visual number discrimination relies on the ability to individuate objects, it is unclear why infants' number discrimination abilities in the visual and auditory modalities are subject to the same thresholds. Similarly, based on their hypothesis, one would expect that visual number discrimination should be improved if visual stimuli are presented sequentially and visual object individuation is no longer an obstacle. However, 6-month-olds are unable to discriminate a unimodal visual sequence when numbers differ by a 2:3 ratio, similar to what is found for simultaneously presented visual stimuli (Jordan et al. 2008b).

Second, infants are able to match numerical information across sensory modalities from birth (Feigenson 2011; Izard et al. 2009). For example, newborns look longer at a visual stimulus that contains the same number of objects as a sequence of tones they hear (Izard et al. 2009). Similar to the ratio-dependent discrimination observed with unimodal stimuli, newborns' cross-modal matching of numerical information is ratio dependent; that is, they are able to match the number of sounds to the right number of objects when the correct and incorrect numbers of objects differ by a 1:3 ratio, but not when they differ by a 1:2 ratio. These findings suggest that infants are able to match numerical information across sensory modalities and across simultaneous and sequential presentation formats. If infants were only able to represent continuous magnitudes as hypothesized by Leibovich et al., it is unclear which continuous dimensions infants would match across sensory modalities and how they would do so without a reference point to determine which value in a given dimension is more or less. For example, should the average tone duration be matched to an average object size, and if so, which duration should correspond to which size? Without familiarization to a range of tone durations and object sizes, it seems impossible for infants to create the reference frame that is necessary to match continuous dimensions. Relying on number is the most parsimonious explanation for the observed patterns in infants' behavior.

Third, infants show greater facility in discriminating number than continuous magnitudes. For example, 6-month-old infants need a 1:4 ratio difference to detect a change in cumulative surface area and a 1:3 ratio difference to detect a change in cumulative perimeter, but only a 1:2 ratio difference to detect a change in number (Cordes & Brannon 2008; Starr & Brannon 2015). Furthermore, when a change in cumulative surface area is directly pitted against a change in number, infants prefer to look at the change in number (Libertus et al. 2014). This preference cannot be attributed to detecting a change in individual element sizes as Leibovich et al. argue (see sect. 3) because individual element size changed by the same ratio as change in number and change in cumulative surface area; that is, when a 1:3 ratio change in number was pitted against a 1:3 ratio change in cumulative surface area, the size of individual elements in both cases changed by a 1:3 ratio. When a 1:3 ratio change in number was pitted against a 1:5 ratio change in cumulative surface area, the size of the individual elements changed by 1:3 and 1:5 ratios, respectively. Despite the greater change in individual element size that accompanied the change in cumulative surface area, infants looked significantly longer at the change in number that was accompanied by a smaller change in individual element size. Therefore, changes in individual element size cannot explain why infants would attend more to a change in number than a change in cumulative surface area. The most parsimonious explanation for the observed findings is that infants are more sensitive to changes in number than continuous magnitudes and not vice versa as Leibovich et al. suggest.

Taken together, these three lines of research suggest that it is most parsimonious to assume that the concept of number is present early in development and that its acquisition does not rest on the acquisition of visual object individuation and experiences with correlations between number and continuous magnitude representations.

Right idea, wrong magnitude system

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Abstract: Leibovich et al. claim that number representations are non-existent early in life and that the associations between number and continuous magnitudes reside in stimulus confounds. We challenge both claims – positing, instead, that number is represented independently of continuous magnitudes already in infancy, but is nonetheless more deeply connected to other magnitudes through adulthood than acknowledged by the "sense of magnitude" theory.

Leibovich et al. argue that it is time to reconsider mainstream "number sense" theories, and as an alternative, they propose a "sense of magnitude" theory based on two central claims. The first is that mental representations of number are non-existent early in human life because number is not represented independently of continuous magnitudes prior to experience with language or the development of executive control. The second is that the associations between number and continuous magnitudes reside in stimulus confounds, making it virtually impossible to examine the true nature of magnitude representations, even in adults. Here, we challenge both claims, positing instead that number may be represented independently early in development,
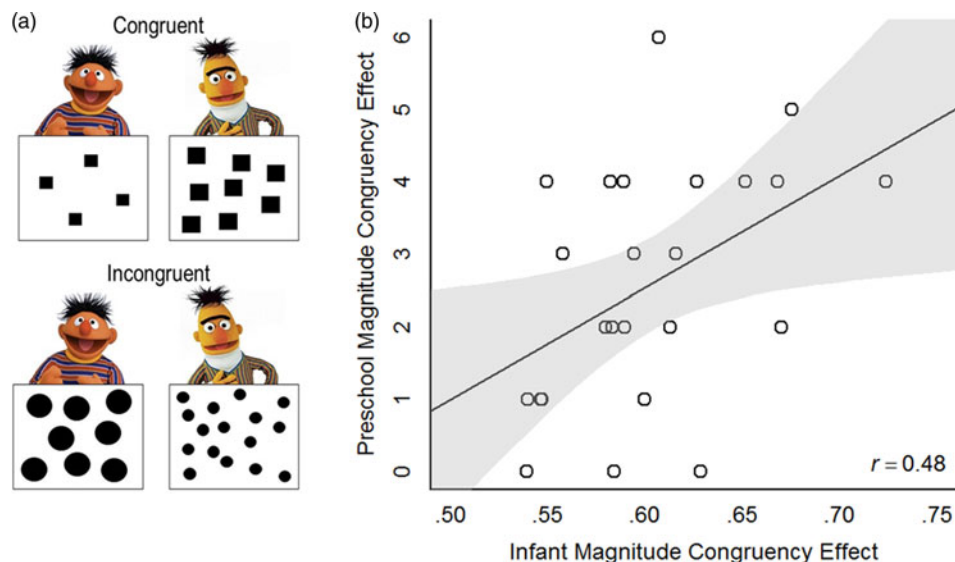

Figure 1. (Lourenco et al.).   (A) Examples of congruent and incongruent trials in the numerical judgment task administered to preschool (3.5-year-old) children in the longitudinal study of Lourenco and Aulet (submitted). Children were asked to judge which character (Bert or Ernie) had the larger numerosity. On the congruent trials, the array with the larger numerosity was also larger in cumulative area and item size. On the incongruent trials, the array with the larger numerosity was smaller in cumulative area and item size. It is worth noting that mean accuracy on the incongruent trials was significantly above chance, confirming the use of number on this task while ensuring an assessment of the influence of area on number. (B) Scatterplot relating magnitude congruency effects across the two time points: infancy and preschool age. For the purpose of this commentary, we depict a subset of children ($N = 25$) whose scores were equal to or above 0.50 at time point 1 (infancy) and 0 at time point 2 (preschool), each of which indicates no association between magnitudes. Infant scores were computed as the proportion of looking time toward the novel test trials as a function of their total looking time (novel+familiar trials; see Lourenco & Longo [2010] for procedural details). Preschool scores were computed as the difference in errors between the two types of trials (incongruent−congruent). Children's performance at 9 months of age was significantly correlated with their performance at 3.5 years of age, suggesting that the association between magnitudes apparent in preverbal children remained relatively stable into preschool age. The scatterplot includes the best-fitting regression line and the 95% confidence interval (shaded area).

but nonetheless shows deep underlying connections with other magnitudes throughout life.

The idea that young infants lack number representations is based on a false premise, namely, that infants are unable to individuate objects. On the contrary, studies of object individuation suggest that by 4 months, infants are skilled at using motion, spatial separation, and featural cues such as shape and size to segment visual scenes, including distinguishing figure from ground, delineating the boundaries of connected or partially occluded objects, and tracking individual objects across time (Atkinson & Braddick 1992; Johnson & Aslin 1995; Kellman & Spelke 1983; Kestenbaum et al. 1987; Needham 1998; Slater et al. 1990; Valenza et al. 2006; Wilcox 1999). Although infants of this age certainly have difficulty with object *identification* (i.e., what an object is [Carey & Xu 2001; Leslie et al. 1998]), object *individuation* independent of identity is well within their capacity. Moreover, our own analysis of visual stimuli used to assess "number sense" in newborn infants challenges Leibovich et al.'s contention that object individuation is impossible because of newborns' poor vision (see sect. 8, e.g., Fig. 6). We calculated spatial frequency (SF) for 50 visual displays using the parameters (e.g., viewing distance: 60 cm) specified by Izard et al. (2009). This analysis yielded a mean SF of 0.5 cycles/degree, well within the normal acuity of a newborn viewing a high-contrast image (Atkinson et al. 1974; Banks & Salapatek 1981; Brown & Yamamoto 1986). Even the most cluttered portions of these displays averaged 1.4 cycles/degree, a value lower than the upper limit of 1.8–2 cycles/degree of newborns' visual acuity.

Leibovich et al. argue that even after infants come to individuate objects, they still cannot differentiate number from continuous magnitudes because this ability depends on linguistic experience (e.g., number words) and executive control (e.g., inhibition). However, this claim is challenged by recent longitudinal data from our lab showing that individual differences in the associations of various magnitudes (number, area, and duration) at 9 months of age predict the extent to which number remains associated with area in the same children at 3.5 years of age (see Fig. 1). The continuity over this period argues against language as a mechanism of differentiation because the children were mostly preverbal at the earliest time point and therefore had not learned any number words. The continuity was also not explained by inter-individual variability in inhibitory control (measured using the Day-Night Stroop task at 3.5 years), arguing against executive control as a mechanism of differentiation. Although we agree with Leibovich et al. (and others) that number words and inhibition may facilitate differentiation of number from other magnitudes (perhaps by drawing attention to individual stimulus items), our longitudinal data suggest that neither factor is necessary, because at least some differentiation is apparent in preverbal infants and cannot be accounted for by inhibition more generally.

Another reason to doubt Leibovich et al.'s claim that number does not dissociate from continuous magnitudes until relatively late in development is that it makes a dubious prediction: Early representations of number should be less precise than those of continuous magnitudes because of children's earlier, and presumably greater, experience with the latter. Multiple studies examining discrimination sensitivity contradict this prediction. For example, Cordes and Brannon (2009; 2011) found that 6-month-olds require a larger difference between sets when discriminating cumulative area than when discriminating number. Similarly, Bonny and Lourenco (in preparation) found that 4- and 6-year-olds' judgments based on cumulative area were no more accurate than those based on numerosity, regardless of the type of display used to assess accuracy (see Fig. 2) and even when stimuli in the number task included continuous magnitudes incongruent with number.

The crux of Leibovich et al.'s sense of magnitude theory rests on the observation that continuous magnitudes available within non-symbolic arrays influence numerosity judgments, even in adults with mature language and executive control. According to this argument, the association between number and continuous magnitudes is merely a by-product of visual cues confounded
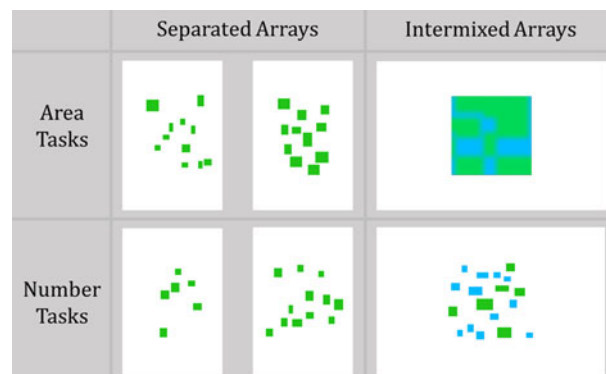

Figure 2. (Lourenco et al.). Examples of stimuli used by Bonny and Lourenco (in preparation) to compare the precision of children's number and area representations. In number and area tasks, children were asked to designate which array contained the greater magnitude. The left column shows displays that were spatially separated, with children selecting either a left or right array. The right column shows intermixed arrays, with children selecting either blue or green as having greater "paint" (area task) or "boxes" (number task). Cumulative area in this study was tested with both discrete (left column) and amorphous (right column) stimuli. Regardless of stimulus type, children never showed an advantage in making area judgments over number, as would be predicted by Leibovich et al.'s theory. Some studies report greater accuracy for area than for number judgments, but "area" in these studies involved single elements (DeWind & Brannon 2012; Piazza et al. 2013) or uniform displays akin to single elements (Odic et al. 2013). When comparing precision, cumulative area is the more appropriate counterpart to number because, like number, it applies to the set of items. Moreover, although infants and children might show greater sensitivity to other continuous magnitudes such as contour length (Cantrell & Smith 2013), the relevant contrast to number, based on Leibovich et al.'s logic, is cumulative area (or perhaps convex hull) because contour length requires object individuation akin to number.

with number in non-symbolic stimuli. However, accumulating behavioral and neural data suggest far deeper connections between number and other magnitudes. For example, even symbolic numbers, for which there are no visual confounds, affect representations of continuous magnitudes: Subliminally primed Arabic numerals bias adults' cumulative area judgments (Lourenco et al. 2016). Crucially, Lourenco et al. ruled out the possibility of priming effects at a decision stage, arguing instead for representations of number and area that partially overlap. Indeed, recent functional magnetic resonance imaging (fMRI) evidence is consistent with overlapping representations in parietal cortex. Harvey et al. (2015) showed not only that number and area share topographic organization, but also that there was a correlation across these maps, with voxels displaying greater activation for larger numerosity also displaying greater activation for items larger in area – a finding that cannot reflect inhibitory processing, as in other neuroimaging studies discussed by Leibovich et al., because the correlation corresponded to a monotonic increase between magnitudes from different displays.

In summary, although we appreciate the prominence given to continuous magnitudes within Leibovich et al.'s sense of magnitude theory, we have argued that disregarding number representations early in development is a weakness (see also Lourenco & Bonny 2016) and that the emphasis on stimulus confounds misses the deeper underlying connections between numerical and non-numerical representations (see also Holmes & Lourenco 2011; Lourenco 2015; 2016). The theory, though a well-intentioned alternative to extant number sense models, suffers from its own limitations.

Infants, animals, and the origins of number

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Abstract: Where do human numerical abilities come from? Leibovich et al. argue against nativist views of numerical development noting limitations in newborns' vision and limitations regarding newborns' ability to individuate objects. I argue that these considerations do not undermine competing nativist views and that Leibovich et al.'s model itself presupposes that infant learners have numerical representations.

Leibovich et al. give two reasons for supposing that humans are not "born with the ability to discriminate numerosities": newborns (1) have poor visual acuity and (2) lack the ability to individuate objects. Their point is that to represent the number of items in a collection, you have to at least be able to see the items and represent each one as being distinct from the others. If newborns lack these minimal capacities, then they would not be in a position to apply numerical representations and would have no need for innate numerical representations. How do infants acquire basic numerical abilities then? According to Leibovich et al.'s model, "number sense develops from understanding the correlation between numerosity and continuous magnitudes" (sect. 8, para. 6). This understanding is grounded in experiences in which infants initially do not distinguish numerosity from continuous magnitudes or distinguish certain continuous magnitudes from others. Aided by exposure to number words, infants come to learn the correlation between number and continuous magnitudes and, eventually, to tease them apart. According to Leibovich et al., children also have to figure out that numerosity and continuous magnitudes do not always correlate (as in Piaget's number conservation task). This knowledge comes later as they learn to inhibit the tendency to form number-relevant judgments on the basis of continuous magnitude. It is only then that children are said to "really understand the concept of numbers" (sect. 8, para. 7).

What counts as really understanding the concept of numbers may be more of a terminological question than a point of substantive disagreement. The interesting question raised by Leibovich et al.'s model is how children come to be able to represent numerical quantities *if they do not start out with some numerical abilities to begin with*. Dehaene (1997) puts the problem vividly and puts his finger on much of the theoretical motivation for nativist accounts of one kind or another: "[I]t seems impossible for an organism that ignores all about numbers to learn to recognize them. It is as if one asked a black-and-white TV to learn about colors!" (pp. 61–62).

Now it is important to keep in mind that nativist and empiricist approaches to explaining the development of numerical abilities occupy opposing regions along a continuum of positions, just like nativist and empiricist approaches to any other representational ability (Margolis & Laurence 2013). On the empiricist side are views that shun innate numerical representations and emphasize domain-general acquisition systems. On the nativist side are views that may include innate numerical representations and that rely on domain-specific systems working in conjunction with domain-general acquisition systems (where domain specificity should be understood as a graded notion). Leibovich et al. associate the nativist view with "the number sense theory," which they take to include a commitment to an innate system for representing number that is automatic, not influenced by continuous magnitude, and realized by distinct neural circuitry. But nativism about numerical abilities need not include these further commitments. For example, a nativist view might postulate an innate numerical system that is realized by neural circuitry that is in close proximity to a system that represents continuous magnitudes, or even an innate numerical system that physically overlaps

with this other system. Such an arrangement would be plausible if the two use similar computations, provide input to common downstream processes, or are a product of an evolutionary history in which one developed out of the other.

What about Leibovich et al.'s claim that newborns should not be expected to possess innate numerical abilities? There are four problems with this claim. First, although newborns cannot see well, that only tells us about their ability to easily apply numerical representations to *visual* stimuli – a limitation regarding the expression of a potential innate representational ability, not a reason to suppose the ability is not there. (Given that newborns have excellent hearing, at best this observation shows that experimentalists should put more effort into tapping newborns' numerical abilities using auditory stimuli.) Second, it is not clear what to make of the claim that newborns cannot individuate items. Leibovich et al. do not explain what the problem is supposed to be – they merely cite Carey (2001) – but if it is that infants do not possess sortal concepts (as claimed in Xu & Carey [1996]), this is not a hindrance to numerical representation. One can still represent numerical quantities (e.g., how many times a lever is pressed or a light flashes) even if one cannot determine whether a cup that appears from behind a screen is distinct from a ball that appears later. Third, whether there is an innate system for representing number (or anything) is not settled by the discovery it is not present at birth (i.e., assuming we take this discovery at face value; but see Izard et al. [2009] and Turati et al. [2013]). Such a system may still require maturation or may be masked by performance factors. Fourth, for this reason, it is helpful to look at the evidence pertaining to nonhuman animals, particularly precocial animals (so that maturation is not an issue) where there can be tight controls in place regarding the experiences that they have prior to testing for numerical representation and where language surely is not driving conceptual development. And there is strong evidence that animals do represent number as such, *pace* Leibovich et al.'s claim that numerical stimuli in this literature are inherently confounded with continuous magnitudes.

Finally, we need to ask about the representational abilities that underpin Leibovich et al.'s model. They claim that children learn about number through experiences that allow them to recognize that numerical properties correlate with continuous magnitudes. But to establish these correlations, one would have to represent the variables being correlated – continuous magnitudes (of different kinds) and number. Rather than explaining where the initial representation of numerical quantity comes from, the model presupposes a certain amount of numerical representation. This may not make the theory identical to the "number sense theory" it opposes, but it does look like Leibovich et al.'s model helps itself to a certain amount of numerical representation, just as nativists claim is necessary for a viable model of numerical conceptual development.

What is the precise role of cognitive control in the development of a sense of number?

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Abstract: The theory put forward by Leibovich et al. of how children acquire a sense of number does not specify the mechanisms through

which cognitive control plays a role in this process. We argue that visual attention and number word knowledge influence each other over development and contribute to the development of the concept of number.

We concur with Leibovich et al.'s central claim that a "sense of number" is acquired, rather than innate. An important question for future research is therefore *how* this sense of number is acquired over development. In their theoretical model of the development of the number concept, the authors proposed that domain-general cognitive abilities, such as language (specifically the acquisition of number words) and cognitive control (particularly inhibitory processes), are important for learning about number. However, they did not specify the mechanisms through which this learning occurs. We argue that considering cognitive control to be domain general is problematic, as control does not operate in isolation, but instead acts in conjunction with relevant domain-specific knowledge that is in and of itself accruing over time (e.g., Amso & Scerif 2015; Johnson 2011). We agree with Leibovich at al. that determining the role of cognitive control in the development of the number concept is complicated by the fact that cognitive control is multi-componential. Here, we focus on two aspects of cognitive control: (1) top-down executive control of attention and (2) bottom-up saliency-driven attention orienting. The two interact with each other, as well as with perception and memory, over development (Amso & Scerif 2015). In this commentary, we propose that the interaction between the development of selective attention to non-symbolic numerosity and the acquisition of the meaning of number words in early childhood contributes to the development of the concept of number.

The theory that non-symbolic representations of numerosity precede, and thus may scaffold the acquisition of, symbolic representations of number has been biased by the assumption that number sense is innate (Feigenson et al. 2004). Leibovich et al.'s rejection of this assumption allows us to consider that the causal mechanism underlying this relationship could go in the opposite direction. Acquiring number knowledge may exert a top-down influence on perception of non-symbolic numerosity. Leibovich et al. mentioned the possibility that learning number words may help children separate discrete numerosity from continuous quantity (Mix et al. 2016). Specifically, "because count words name the property of number, they could be potent attention-directing cues" (Mix et al. 2016, p. 20). Children are typically thought to understand the meaning of number words once they have learned the cardinality principle, which is that the last number word used when counting a set indicates the number of objects in the set. There is evidence that perception of non-symbolic numerosity differs between young children who have acquired the cardinality principle and children who have not (Negen & Sarnecka 2015; Slusser & Sarnecka 2011). Specifically, 2- to 6-year-old children who had not yet acquired the cardinality principle failed to choose the more numerous of two non-symbolic arrays when discrete number conflicted with continuous quantity, whereas children who had acquired the cardinality principle succeeded (Negen & Sarnecka 2015). Furthermore, in another study, 2- to 4-year-olds were asked to sort cards depicting arrays of objects that varied along the dimensions of color, shape, and numerosity, and all children successfully sorted by color and shape. However, only children who knew the cardinality principle accurately sorted cards based on the number of objects in the arrays (Slusser & Sarnecka 2011). This suggests that once children learn the meaning of number words, as evidenced by their grasp of cardinality, they develop a better attentional template for discrete numerosity. However, as the existing evidence is correlational, the direction of the relationship between learning the cardinality of numerical symbols and processing the numerosity of non-symbolic arrays remains unclear.

An alternative to the suggestion that a non-symbolic number sense per se fosters symbolic understanding is that certain

properties of non-symbolic arrays increase the bottom-up saliency of discrete numerosity, and that this could in turn influence the development of understanding the cardinality of number symbols. For example, young children are more likely to use discrete number to make magnitude judgments for sets smaller than four than for larger sets (Cantrell et al. 2015b). Constraints of visual attention could explain this discrepancy in magnitude judgments across set sizes: four is the maximum number of objects that can be attended to in parallel (Trick & Pylylshyn 1994). Thus, arrays of up to four objects can be enumerated quickly and accurately, known as *subitizing* (Kaufman et al. 1949). Leibovich et al. argued that subitizing was not relevant to their theoretical proposal as most studies discussed focused on non-symbolic magnitudes outside of the subitizing range. However, they ignored the possibility that the ability to attend to individual items in parallel may play a role in learning to select discrete numerosity separately from non-numerical magnitude. Indeed, Carey (2001) proposed that children map number words up to four onto representations of non-symbolic quantity of small sets. Therefore, bottom-up attention to numerosity could enable learning the initial meaning of number words.

A third, and we believe more likely, possibility is that that an interaction between bottom-up and top-down attention (guided by relevant number knowledge) scaffolds children's acquisition of the number concept. Such a perspective could resolve the debate over whether efficient processing of approximate non-symbolic number is a cause or a consequence of the understanding of cardinality. Slusser and Sarnecka suggested, "children's understanding of the cardinality principle […] *is the same thing as* their understanding that number words pick out numerosities" (2011, p. 9). Therefore, once children understand that the word *five* refers specifically to five objects, rather than the overall area they subtend, they have learned to select numerosity as a relevant stimulus dimension and come to understand that two sets of five objects are related based on their numerosity. This suggests that the salience of numerosity does not precede the ability to voluntarily select it, but rather that they influence each other bi-directionally.

In conclusion, future research should investigate learning mechanisms underlying the acquisition of a sense of number. Rather than studying cognitive processes in isolation, interactions between perception, attention, and number knowledge should be investigated. Furthermore, gaining a better understanding of developmental learning mechanisms necessitates testing bi-directional hypotheses and moving beyond correlational, cross-sectional approaches.

Commentary on Leibovich et al.: What next?

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Abstract: The conclusions reached by Leibovich et al. urge the field to regroup and consider new ways of conceptualizing quantitative development. We suggest three potential directions for new research that follow from the authors' extensive review, as well as building on the common ground we can take from decades of research in this area.

The question of whether human quantitative abilities are rooted in an early emerging (if not innate) sense of discrete number has defined the research agenda in this area for 30 years. This superb piece of scholarship by Leibovich et al. brings the

ongoing debate into sharp focus. Their conclusion, based on a thorough review of behavioral and neural evidence, is that there is no clear indication of strictly numerical processing, and indeed, it may be impossible to isolate number from its quantitative correlates. This is a stark conclusion given the intense efforts directed toward developing an account of innate number processing; however, we agree that the evidence seems to compel this conclusion. Mix et al. (2002a; 2002b) raised similar concerns nearly 15 years ago, and research generated in the intervening period has done little to alter this view. We agree with the authors that the time has come to move on to different questions, but what questions might these be?

To begin, let us take stock of what we know. It would be wrong to dismiss the innovative literature that attempted to disentangle number from its correlates. Though its explicit aims may be unattainable, this body of work nonetheless contributed a large corpus of findings on which to build. It reveals, for example, that human beings are sensitive to quantitative information from a very early age – perhaps as early as the first hours of life (Antell & Keating 1983). This fact is so commonly accepted that it may seem unimpressive. However, immediate apprehension of quantity at birth was not the only plausible developmental scenario. One might have predicted infants need months of exposure to different quantities before they notice subtle distinctions, just as they seem to need months of exposure to detect categories (e.g., Rakison & Poulin-Dubois 2001). It is actually rather astonishing that human beings come into the world prepared to process quantitative information much as they seem prepared to process physical properties like solidity and support (Baillargeon 1994) or basic percepts like color and shape (Ricciuti 1965). Without the pioneering research on preferential looking and habituation, we might not know this. Second, there is general agreement that infants and young children – in fact, people of all ages as well as some animals – process large sets differently than they process small sets (i.e., ≤3). Though this was first demonstrated in adults in the late nineteenth century (cf. Mandler & Shebo 1982), there was virtually no evidence in children prior to the late 1970s. The notion that large sets are processed in a ratio-dependent manner was well known in the psychophysical literature, but the application of these ideas to quantitative perception, particularly among non-verbal infants, is a powerful advance. Finally, to our knowledge, no one has ever claimed that infants perceive discrete number to the exclusion of other quantitative cues. There is a long tradition of research examining infants' sensitivity to continuous dimensions of quantity, such as complexity, density, surface area, and brightness (e.g., Gao et al. 2000; see Mix et al. 2016 for a review) and despite attempting to isolate number from these percepts, there has been no argument about whether infants can use these alternate cues. It is widely accepted that they can, and this is an important point of agreement.

The main point of contention has been whether infants and young children isolate discrete number from a perceptual stream that obviously includes other cues. As Leibovich et al.'s review elegantly demonstrates, this question has not only gone unanswered, but also may be unanswerable. However, rather than sounding the death knell of research on early number, this conclusion raises many equally important questions that have been largely overlooked. One question is how infants might profit from the natural correlations among quantitative variables to discover how these relations work. Cantrell and Smith (2013) proposed that numerical representations are strongest and most likely to be noticed when quantitative cues are correlated (e.g., more items also have more contour, more area, more complexity, etc.). When these dimensions vary randomly, the number representations themselves are more fragile and less likely to sustain attention. On this view, infants might discover number by noticing more intense perceptual

stimulation and eventually realizing this intensity is correlated with other percepts. Indeed, when the consistency of these correlations was varied, infants demonstrated greater acuity for both number and area when the correlations were maximally correlated (Cantrell et al. 2015a). This line of inquiry demonstrates the insight gained by viewing quantitative perception as a complex system of interrelated signals, rather than a collection of isolated streams. Another important direction for future research is to coordinate the literatures on object individuation, object tracking, short-term memory, and number perception. These processes are inherently interrelated. One cannot perceive number without separating items to be enumerated and either tracking or remembering them (Mix et al. 2002a; 2002b; 2016). Likewise, keeping track of several separate objects is tantamount to representing number (Xu & Carey 1996). Though experiments on one of these topics usually incorporate stimulus variations from the others (e.g., Ross-Sheehy et al. 2003), the developmental relations among these processes have not been fully explored. Yet, understanding these relations may be critical as unitization could be a major mechanism for isolating number (Mix et al. 2016). Related to this, the role of verbal counting in early quantification is ripe for re-examination as count words may provide the attentional scaffolding that allows children to inhibit responses to irrelevant quantitative cues. Studies on the developmental relations involving number sense and verbal number acquisition have begun to amass (see Merkley & Ansari 2016 for a review) and indicate that knowledge of the count words leads to improved acuity for perceptual quantification. More studies are needed. For example, researchers might ask whether unitization emerges before or after acquiring verbal count words.

In summary, Leibovich et al.'s conclusions urge the field to regroup and consider novel ways of conceptualizing quantitative development. We are excited to see where these new directions take us.

Number faculty is alive and kicking: On number discriminations and number neurons

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Abstract: Leibovich et al. advocate for a single "sense of magnitude" to which a dedicated faculty for number could allegedly be reduced. This conclusion is unjustified as the authors adopt an unnecessarily narrow definition of "number sense," neglect studies that demonstrate non-symbolic numerosity representation, and furthermore ignore abstract number representations in the brain.

The idea that discrete-numerical quantity and basic continuous magnitudes share some characteristics is uncontested. However, in their recent article, Leibovich et al. propose that a separate faculty for number does not exist in the first place. The article is, however, rather selective, as the authors reach this conclusion based on a survey that is idiosyncratic not only in what it attacks, but also in what it omits.

First, they adopt an unjustifiably narrow definition of "number sense." The term "number sense" was first coined by Tobias Dantzig (1930). However, he is nowhere mentioned in the target article. Dantzig (1930, p. 1) writes, "Man, even in the lower stages of development, possesses a faculty which, for want of a better name, I shall call Number Sense." Although Dantzig

only talks about a faculty to assess numerical quantity, the target article conveys a rather simplistic view of an "innate" and fully hardwired system that extracts numerical information just like a reflex. However, such a narrow definition does not hold for any cognitive capability and is not maintained by protagonists of the "number sense" (Anobile et al. 2016c; Burr & Ross 2008; Viswanathan & Nieder 2013). Physiological faculties are plastic (subject to maturation and/or learning processes); they are embedded in – and interact with – other faculties. The finding that the number faculty interacts with general magnitude representations can therefore not refute its existence.

Second, a key argument of the article is that varying the number of items in a set inevitably changes physical stimulus parameters. Although this is undisputed, it is far too premature to conclude that investigations of numerical representations are therefore a priori useless. The two main reasons are as follows:

1. Potential sensitivity to simple sensory parameters is not specific to number investigations but pervasive to all investigations targeting abstract representations. Semantic groups can only be tested with specific stimulus representatives. Continuous magnitudes are, of course, no different in that respect. Resorting to continuous magnitude therefore does not solve the problem.

2. In contrast to the impression caused by omissions in the target article, many researchers painstakingly selected their stimuli and went to great lengths to demonstrate number representations. Because it is not physically possible to equate all possible stimulus parameters at the same time, the best way is to control – unbeknown to the subject – one parameter after the other in separate stimulus configurations. If the subject responds equally to systematically varied numerosity stimuli, it is safe to conclude that the subject responds to number (Nieder 2016). One of the main research agendas over the last two decades therefore was to test numerosity representations over a broad range of stimuli and formats. For example, humans have recently been shown to be far more sensitive to numerosity than to continuous magnitudes in dot displays (Cicchini et al. 2016). Greater sensitivity to changes in numerosity was present both spontaneously and in tasks where participants were explicitly instructed to judge continuous parameters of the dot displays. Therefore, humans extract number information based on dedicated mechanisms. In addition, studies using controlled stimuli with conditioned animals demonstrated clear numerosity judgments. For example, the seminal monkey study by Brannon and Terrace (1998) controlled for item location, overall surface area, item size, and item type. Later, Nieder et al. (2002) controlled for item position, overall item area, overall item circumference, high and low density, item type, shape-like item configurations, and linear item arrangements, the latter one also abolishing convex hull. Monkeys also extracted the number of elements that appeared sequentially one-by-one and matched it to the number in spatial dot arrays (Nieder et al. 2006). In this sequential presentation format, temporal parameters such as duration, rhythm, and accumulated intensity have been controlled for and were neglected by the monkeys. Moreover, monkeys assessed number also independently of the sensory modality and discriminated both the number of sequential visual dots and auditory sounds within the same session (Jordan et al. 2008a; Nieder 2012). The animals did not care about non-numerical magnitude changes and responded to number information. Similar results have been obtained in preschool children (Barth et al. 2005). In sum, evidence for the capability of nonverbal subjects to represent numerical quantity is stronger than ever.

Third, another unfortunate omission of Leibovich et al. concerns abstract number representations in the brain. The single-neuron code underlying number representations has been addressed over the past years with a broad range of controlled stimuli. These studies in animals showed surprisingly abstract number representations ("number neurons"). As reviewed in Nieder (2016), number neurons recorded in monkeys performing the aforementioned numerical tasks were tuned to preferred numerosities while being largely insensitive to changing sensory features. Number neuron responses in prefrontal cortex (PFC), and to some extent in the intraparietal sulcus (IPS), generalized across spatial features in visual item arrays (Nieder et al. 2002), spatio-temporal visual presentation formats (Nieder et al. 2006), and also visuo-auditory presentation formats to signal numerosity supramodally (Nieder 2012). Moreover, in monkeys trained to associate shapes with numerosities, neurons signaled the numerical meaning of signs (Diester & Nieder 2007). Number neurons were present even if monkeys were not trained on number (Viswanathan & Nieder 2013). After training, PFC showed improved responses to numerosity during active discrimination, whereas ventral intraparietal area (VIP) neurons remained stable (Viswanathan & Nieder 2015). Of course, such highly generalized responses of number neurons cannot (and should not) be expected to be the only code for numerical quantity. Abstract number information can also be extracted from population activity (Ramirez-Cardenas et al. 2016; Tudusciuc & Nieder 2007). Collectively, these single-neuron recordings strongly support the idea of a dedicated number faculty residing in a parieto-frontal network, with striking similarities between numerical representations in nonhuman and human primates (Nieder 2016).

Leibovich et al. also err when claiming that only one study (Castelli et al. 2006) had directly compared brain areas during number and continuous magnitude comparison tasks. For example, Pinel et al. (2004) found that number and size, but not luminance, activated overlapping parietal regions during functional imaging. More directly, single-cell recordings in monkeys that discriminated numerical, spatial, and sensory magnitudes in one session showed that coding was largely dissociated at the single-neuron level (Eiselt & Nieder 2016; Tudusciuc & Nieder 2009). Therefore, numerical representations are based on distributed coding by single neurons that are anatomically intermingled within the same cortical area.

Contrary to the claim of the target article, overwhelming evidence supports a dedicated number faculty that operates independent from continuous magnitude. The target article's attempt to reduce number judgments to simple magnitude representations is a lost case. Far from being put to rest, the number faculty is alive and kicking.

The contributions of non-numeric dimensions to number encoding, representations, and decision-making factors

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Abstract: Leibovich et al. suggest that congruency effects in number perception (biases towards smaller, denser, etc., dots) are evidence for the number's dependence on these dimensions. I argue that they fail to differentiate between effects at three distinct levels of number perception – encoding, representations, and decision making – and that differentiating between these allows the number to be independent from, but correlated with, non-numeric dimensions.

Visual and auditory number stimuli inherently correlate with dimensions such as size, density, rate, and so forth, and observers are sometimes biased towards these dimensions: Changing the density of a collection of dots also changes which set observers believe to be more numerous. Leibovich et al., following the footsteps of recent findings reporting such "congruency effects,"

argue that number may be entirely or partly dependent on these dimensions – that there is no innate number sense independent from our perception of density, convex hull, size, and so forth.

However, their critique leaves a key question open: At what level of processing do non-numeric dimensions exert their hold on number? There are at least three independent possibilities, and only one of them is consistent with the central claim against an independent number sense.

The first possibility is that number is encoded using low-level visual features, such as orientation, contrast, spatial frequency, and so forth, which are shared with other dimensions, rather than out of its own dedicated feature detectors (e.g., Dakin et al. 2011 vs. Burr & Ross 2008). For example, consider that face perception strongly depends on a unique mix of low- and high-spatial frequency, and, therefore, changing frequency information also changes which emotion is most strongly perceived (Vuilleumier et al. 2003). In this same manner, there are now many reasons to suspect that number encoding depends on features such as low-spatial frequency (Dakin et al. 2011), and that it may even depend on distinct features at different levels of crowding (Anobile et al. 2014). Thus, manipulating density (i.e., low-spatial frequency information) can result in changes in number perception, not because of number being represented *as* density, but rather because of their shared dependence on identical low-level features. Congruency effects, therefore, could be interpreted as positive results describing the nature of low-level features used to encode number, not as evidence against its dependence on non-numeric dimensions. At the very least, claims to number's non-independence must first account for the shared low-level features.

The second possibility is that number and non-numeric dimensions compete for the same decision-making component, such as putting a common load on working memory or yielding similar response conflicts (Hurewitz et al. 2006; Odic et al. 2016; Van Opstal & Verguts 2013). Once again, consider an analogy: congruency effects found in the Stroop effect do not imply that color perception is dependent on and statistically learned from reading ability, but rather that multiple dimensions can compete for the same response. Because we know that density and area perception tend to be more accurate in adults compared with number, there is plenty of reason to think that these dimensions will win a "horse race" for the same response as number, creating congruency effects without any shared representations (Hurewitz et al. 2006). Consistent with this, my colleagues and I have demonstrated that number and time perception only correlate when individual differences in working memory are not controlled for (Odic et al. 2016). More recently, we have found that the effect of non-numeric dimensions such as contour length is entirely eliminated when Stroop-like response conflicts are alleviated (Picon & Odic, in preparation). Together, these results suggest that many demonstrated congruency effects could be response conflicts, and that any claim for dependence between number perception and non-numeric dimensions should first control for these factors.

Finally, the third possibility for the link between number and non-numeric dimensions – and one that is most consistent with the claims of Leibovich et al. – is that number may be (antecedently) represented on the same representational scale as other dimensions, either by being directly represented as, for example, area, or alternatively by being represented on a domain-general, unitless magnitude scale that simply codes for *more* versus *less* (Cantrell & Smith 2013; Lourenco & Longo 2010; Walsh 2003). Although Leibovich et al. suggest that statistical learning eventually separates number from these dimensions, their theory requires that – from birth until some later age – numerical information is represented in one of these two ways. But, as reviewed previously, evidence for shared representations must first control for the possibility of shared encoding or decision-making factors; given that the majority of existing work fails to do so, what is the evidence for shared/unified representations? Perhaps the most convincing case cited by Lebovich et al. is that of

Tudusciuc and Nieder (2007), who found neurons in the parietal cortex that respond to both number and length. But a closer inspection of their data reveals that these neurons often code in opposing ways: The same neuron may code for small numbers, but very long lengths, or vice versa, running contrary to the idea of a shared scale and instead consistent with a set of overlapping population coding neurons that play different roles for each dimension.

Another approach at demonstrating shared representations is to simultaneously measure number and the candidate shared dimensions, such as area, length, density, and time; if number shares the scale for these representations, any individual and developmental variability within number should be accounted for by differences in these other dimensions. Recently, my lab followed this logic through and tested 2- to 12-year-old children and adults on these five discrimination tasks. We found that number develops independently from area, length, density, and time, which in turn develop independently from it stretching back to age 2 (Odic 2017; see also Odic et al. 2013). Hence, unless the kind of proposed statistical learning proposed leads to complete differentiation by age 2, it is difficult to imagine how these results could be obtained without a significant independence between number and area, length, density, and time perception.

To conclude, Leibovich et al. make a bold claim – that congruency effects are illustrative of number's dependence on non-numeric dimensions – but their critique fails to account for the possibility that these effects stem from shared encoding or decision-making components, not shared representations. Future work exploring number's dependence should carefully disentangle the contributions of other dimensions to encoding and decision making, as these levels are not constitutive of the independent representations of number.

Numerical magnitude evaluation as a foundation for decision making

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Abstract: The evaluation of magnitudes serves as a foundation not only for numerical and mathematical cognition, but also for decision making. Recent theoretical developments and empirical studies have linked numerical magnitude evaluation to a wide variety of core phenomena in decision making and challenge the idea that preferences are driven by an innate, universal, and stable sense of number or value.

Leibovich et al.'s critique of the "number sense" theory is timely and has implications beyond the literature on numerical and mathematical cognition. Numerical magnitude perception also plays a critical role in decision making, as it shapes how people trade off outcomes that vary in size, probability, and timing. Moreover, recent theoretical developments and empirical findings from the study of decision making have shown that evaluations of numerical magnitudes are neither innate, nor universal, nor stable, but vary substantially across countries, individuals, and contexts.

The evaluation of numerical magnitudes in decision making. The evaluation of numerical magnitudes is critical to decision making and implicitly forms the core of many important theories of choice (e.g., Kahneman & Tversky 1979). For example, an individual faced with multiple job offers needs to compare (among other

things) their salaries and assign a subjective value or "utility" to each of these monetary amounts before she can make a proper decision. Similarly, in deciding whether to purchase a good, how much of the good to purchase, or how much we are willing to pay for that good, we must be able to assign subjective "disutilities" to the monetary costs of paying for our purchases, to see whether they outweigh the benefits we expect to receive from those purchases. It turns out that subjective evaluations of monetary gains and losses resemble those for numbers and most perceptual magnitudes: People are much more sensitive to differences between small financial gains (or losses) than they are to the same differences between larger gains (or losses) (Kahneman & Tversky 1979; Tversky & Kahneman 1992).

This problem of assigning subjective values (or utilities) to numerical decision outcomes applies to many different types of choice attributes, and not just monetary ones like salaries and prices. For example, a restaurant patron may wish to evaluate the number of calories associated with each item on the menu, whereas a policy maker considering measures to increase road safety needs to evaluate their potential reductions in human fatalities. As with financial gains and losses, subjective evaluations of non-monetary outcomes tend to exhibit diminishing sensitivity (e.g., Olivola 2015; Slovic 2007). Moreover, the role of numerical magnitude evaluation in decision making extends beyond assigning (dis)utilities to the outcomes themselves, as one also needs to consider their likelihoods of occurrence and when (in time) they are expected to occur. Specifically, uncertain or delayed outcomes need to be discounted relative to certain and immediate ones. Research suggests that numerical probabilities are typically transformed non-linearly into an intuitive sense of likelihood (Prelec 1998; Tversky & Kahneman 1992; Wu & Gonzalez 1996). Similarly, research suggests that people tend to evaluate time delays in a non-linear fashion (Frederick et al. 2002; Olivola & Wang 2016; Read 2004).

Recent theoretical and empirical developments. Although there have been continuous efforts by decision-making researchers to map the relationships between choice-relevant numerical magnitudes and their subjective values or weights, most of this research has operated separately from the field of numerical cognition, and very little of it has examined *why* or *how* the observed mappings occur (Olivola & Chater 2017). Fortunately, the last decade has witnessed a growing interest in understanding how the evaluation of numerical magnitudes relates to decision making, and this has led to several important insights and theoretical advances. We have learned, for example, that individual differences in symbolic-number mapping predict how people value monetary outcomes (Schley & Peters 2014), and that individual differences in subjective perceptions of temporal distance (i.e., how "far away" a given time delay seems) predict how patient people will be when making intertemporal trade-offs (Kim & Zauberman 2009; Zauberman et al. 2009). Some researchers have also proposed a novel theory of decision making that explicitly attempts to explain the magnitude evaluation process that underlies outcome valuation, probability weighting, and time discounting (Stewart 2009; Stewart et al. 2006; see also Kornienko 2013). According to this "decision by sampling" theory, people evaluate monetary and non-monetary outcomes, probabilities, and time delays by comparing them with other relevant values stored in memory. For example, an individual would determine the value of a particular financial gain (e.g., receiving $100) by comparing it with a sample of other financial gains that she has previously experienced or observed. Depending on the composition of her memory sample, the target numerical magnitude being evaluated will either seem large (if it ranks higher than most comparison values), small (if it ranks lower than most), or of medium size (if it ranks close to the median value). It turns out that this theory successfully explains a wide variety of core phenomena in decision making, such as reflective risk preferences (Stewart & Simpson 2008; Stewart et al. 2006), loss aversion (Olivola & Sagara 2009; Stewart et al. 2006; Walasek & Stewart 2015), non-linear

probability weighting (Stewart et al. 2006), hyperbolic discounting (Stewart et al. 2006), and the diminishing sensitivity to human fatalities (Olivola & Sagara 2009; Olivola et al. 2017). In doing so, it explicitly connects the process of magnitude evaluation to many important preference patterns.

Evaluations of choice-relevant magnitudes are neither innate nor stable. The research we have discussed also highlights the malleability of choice-relevant magnitude evaluations and thereby casts doubt on the idea that decisions are driven by an innate, universal, and stable sense of number or value. Individuals differ in their perceptions of numbers and time delays, and these differences reliably predict their risk and time preferences (Kim & Zauberman 2009; Schley & Peters 2014; Zauberman et al. 2009). Some of these individual differences are likely the result of variations in people's experiences (Ungemach et al. 2011), which in turn reflect differing environments (Olivola & Sagara 2009). Consequently, individuals from different countries may perceive and respond very differently to similar outcome magnitudes (Olivola & Sagara 2009). In fact, choice-relevant magnitude evaluations can even vary within individuals if the distribution of comparison values changes (Olivola & Sagara 2009; Ungemach et al. 2011; Walasek & Stewart 2015). In sum, these decision-making findings suggest a highly individual- and context-dependent evaluation of numerical magnitudes, rather than a universal and stable "number sense."

How not to develop a sense of number

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Abstract: The authors rightly point to the theoretical importance of interactions of space and number through the life span, yet propose a theory with several weaknesses. In addition to proclaiming itself unfalsifiable, its stage-like format and emphasis on the role of selective attention are at odds with what is known about the development of spatial-numerical associations in infancy.

The mechanism of numerosity perception is often depicted as a solitary creature, living alone in the intraparietal sulcus (IPS) (its presumed birthplace), with nothing to feed it but number, and serving just the function for which it was born – giving the rest of the brain an approximation of "how many." Although copies of this mechanism are thought to exist in the brains of an astonishing range of organisms – even those with no homologue to IPS – it is consistently predicted to have the same basic properties (noise and ratio dependence) everywhere one finds it, regardless of its age, the genes in its cells, or its history of activity.

This portrait of the numerosity mechanism – only slightly exaggerated – is in dire need of revision, and we thank Leibovich et al. for pointing toward a major issue for any future theory. As the authors note, our perception of numerosity is influenced by the spatial characteristics of the set (Frith & Frith 1972; Gebuis et al. 2014). Similarly, our perception of space is influenced by numerical aspects of the set (de Hevia et al. 2008). Two

major – but still unsettled – issues arise from these findings. The simpler issue is describing the relation between two groups of processors: those that register the number of a set (numerosity detectors) and those that register the non-numerical spatial characteristics of the set (e.g., spatial frequency). The harder issue is to *explain* how this relation does or does not change over time (for a review of evidence, see McCrink & Opfer 2014); this is the purpose of a developmental theory.

Although the authors have succeeded in describing the rich interactions that exist between space and number, the cornerstone of their article – their developmental theory – suffers from three major weaknesses.

First and foremost, the argument is structurally flawed. According to the authors, the number of items in a set and the non-numerical spatial properties of a set are so correlated that it is "impossible" (sect. 5.1, para. 3), "nearly impossible" (abstract), or at least really hard to tell (the authors seem to be of more than one mind on this issue) whether judgments of numerical magnitude are judgments about number or non-numerical correlates of number. Logically, then, the very existence of a stage in which organisms cannot distinguish number from magnitude is unfalsifiable. Also unfalsifiable is the very existence of a stage in which subjects can distinguish number from magnitude. This is a serious weakness. A theory that begins by pronouncing itself unfalsifiable is a non-starter.

The second challenge to their developmental theory involves the role of inhibitory control and correlational learning. "Number sense," we are told, "develops from understanding the correlation between numerosity and continuous magnitudes" (sect. 8, para. 6). Inhibitory control then allows children to ignore irrelevant continuous magnitudes, with number words aiding this inhibitory process by emphasizing the cardinality of a set over continuous magnitudes. But this argument has a built-in contradiction: If children do not already have a sense of number, what exactly are they inhibiting when number and non-numerical magnitude conflict? The same issue arises for correlational learning. If children do not already have a sense of number, how can they track the correlation between number and continuous magnitude? Logically, number must be perceived *before* learning to select numerical over non-numerical cues and *before* learning to track what correlates with number. Empirically, this is also what the evidence indicates. Of the nine cross-modal mapping studies reviewed by Cantrell and Smith (2013), six of them found evidence of cross-modal mapping in infants (who notoriously lack inhibitory control). Therefore, we agree that inhibition and correlational learning improves the quality of numerical comparison, but their causal argument for the developmental sequence is logically untenable.

The final challenge for the developmental theory comes from the collapse of stage theories in general. Like all stage theories, the authors' theory depicts the development of numerical competence as proceeding in an invariant sequence of broadly applicable, age-related achievements. Empirically, development is seldom this orderly. Against Piaget's theory, for example, children who appeared "pre-operational" using one conservation task were found to conserve just fine on another, and the types of errors that a child would make on one conservation task seldom appeared later on the same task or on different tasks (Siegler 1981).

Do the stages proposed by the authors fare any better than Piaget's stages? We think not. According to their theory, children with normal visual acuity learn to correlate number with continuous magnitude only after they represent number, which occurs when they learn number words. If so, one would not expect pre-linguistic infants with normal vision to associate numerosity with magnitude, because that would violate the order of the stages. This idea has been tested directly in pre-linguistic infants with visual acuity near adult levels (de Hevia & Spelke 2010; Lourenco & Longo 2010). In one such experiment, Lourenco and Longo (2010) found that

9-month-old infants easily learned an arbitrary numeric rule (less numerous sets of 2 objects are white, more numerous sets of 4 objects are black) and generalized this rule to new sets (e.g., sets of 5 and 10). Critically, infants generalized the rule to sets of a new *size* as well. Moreover, infants who were given a size-based rule at habituation generalize the learned rule to sets of a discrete number. Thus, pre-linguistic infants, who are not supposed to be in the stage where they can learn this sort of thing, applied a learned rule involving "more than" and "less than" across spatial and numerical dimensions, even when trained in only one dimension. These results contradict the authors' stage theory. However, they accord with a developmentally continuous theory that infants have spatial-numeric associations that arise for reasons (such as partially overlapping neural architecture) that have nothing to do with visual acuity.

In summary, although we applaud the authors for bringing attention to the findings of spatial-numeric associations in early development, we do not think the field needs another unfalsifiable and logically contradictory stage theory.

Direct and rapid encoding of numerosity in the visual stream

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Abstract: The target article dismisses all prior work purporting to demonstrate that number is a conceptual primitive. Here, we take issue with their misrepresentation of our recent line of work on numerosity perception, which demonstrates rapid and direct encoding of numerosity and undermines the thesis of the target article that "continuous magnitudes are more automatic and basic than numerosities" (sect. 1, para. 2).

Using a novel stimulus set and regression approach, we recently demonstrated that participants primarily use number rather than other visual features in a numerical discrimination task (DeWind et al. 2015). Combining this approach with high-temporal-resolution electroencephalography (EEG) recording, we further demonstrated that numerosity has a larger effect on very early neural activity than other non-numerical magnitudes (Park et al. 2016b). The target article mischaracterizes these methodological advances and overlooks the main findings of this work, which support early and direct access to visual number.

A novel regression approach and behavioral primacy of number. Non-numerical features of an item array are necessarily partially correlated with number. Nevertheless, as demonstrated in DeWind et al. (2015), an appropriately designed stimulus set can vary each feature such that they are *partially* independent from number and from each other and can be reduced to a common basis set (number, size, and spacing). As with any regression analysis, our approach leverages the unique variance of each feature to determine which feature affects the variance in a dependent measure, whether it is behavior (DeWind et al. 2015) or neural activity (Park et al. 2016b). In particular, that regression approach provides a way to test the degree to which participants use non-numerical response strategies in a numerical comparison task. Critically, participants made comparisons on the basis of number, and much more so on number than other visual features (DeWind et al. 2015).

Passive viewing paradigm. Numerosity judgments are influenced by non-numerical cues and vice versa, and such influences are non-linear and asymmetric (Allik & Tuulmets 1993; Gebuis & Reynvoet 2012b; Ginsburg & Nicholls 1988; Miller & Baker 1968; Mix et al. 2002a; Nys & Content 2012; Soltesz et al. 2010; Sophian & Chu 2008; Tokita & Ishiguchi 2013). Thus, the representation of one dimension (say, numerosity) will differ depending on whether the task requires focusing on numerosity or some other dimension. One can circumvent this issue by using a neural approach with a passive viewing paradigm. Indeed, using EEG to measure neural activity while participants were passively viewing dot arrays, we tested which magnitude dimension most contributes to the modulation of neural activity in a task-free design (Park et al. 2016b). However, the target article incorrectly states "a strong correlation between number and continuous magnitudes can change strategy [in our study]" (sect. 6, para. 2), when in fact there was no strategy involved.

Even under a passive viewing paradigm, attention might be directed toward one feature dimension over another because a larger range in one stimulus dimension may increase salience and consequently override the effects of another dimension with a smaller range. For an extreme example, imagine the apparent contrast of a set of 10 dots each with a radius of 1 cm and a set of 11 dots each with a radius of 0.1 cm. Clearly, the salience of the area difference would overwhelm the number difference, and neural activity modulated by such large salient differences in area could easily mask neural activity modulated by relatively less salient number differences. Therefore, it is important to use the same range of values in each magnitude dimension for a fair comparison between them. The target article incorrectly states that Park et al. (2016b) used a greater range of continuous magnitudes, when across two experiments we indeed used dot arrays that were constructed to cover equal ranges of number, size, spacing, total area, item area, total perimeter, item perimeter, convex hull, density, coverage, and overall scale. In fact, in Park et al. (2016b), we made this exact critique of the experimental design used by Gebuis and Reynvoet (2013), which employed a larger range of continuous magnitudes than numerosity. The Leibovich et al. (2016b) article that the authors rely on to develop their thesis suffers from this criticism because there was a greater difference in non-numerical magnitudes (ratio of about 2:5) than in numerical magnitudes (ratio of about 3:5). Therefore, the observed smaller interference of numerosity in non-numerical magnitude compared with the reverse in that study could have been driven by differences in the ratios between the two dimensions. Collectively, the target article mischaracterizes the stimulus design in Park et al. (2016b) and fails to recognize the implications of having unequal magnitude ranges in the very studies that it relies on to build the main thesis (e.g., Gebuis & Reynvoet 2013; Leibovich et al. 2016b).

High-temporal-resolution recording of neural activity. The target article asks which magnitude dimension is more "basic, innate, and automatic" (sect. 5.1, para. 3). In fact, the main contribution of our event-related potential approach (in combination with the aforementioned stimulus design and regression approach) was the demonstration of selective neural sensitivity to numerosity very early in the visual stream, prior to any neural sensitivity to other non-numerical magnitudes (Park et al. 2016b). Such a robust and selective effect of numerosity with negligible effects of non-numerical magnitudes was demonstrated in two independent experiments in Park et al. (2016b) and is now replicated in similar experiments investigating different neural index and different ranges of numerosities (Fornaciai & Park 2017; Park 2017). These results directly contradict the authors' conclusion that the representation of numerical magnitude stems from continuous magnitudes. Instead, our findings support the idea that numerosity is perceived directly and rapidly in the visual stream.

Conclusion. The target article argues that all prior studies suffer from flaws such that any claim of pure numerical judgments or numerical selectivity in the brain could be attributed to a generalized magnitude system. However, for the reasons mentioned previously, we find the authors' coverage of the prior work addressing these issues problematic and find their case for dismissing evidence that number is a salient primitive far from convincing. Moreover, at least 10 different continuous magnitude dimensions can be uniquely defined (see DeWind et al. 2015), but the target article lacks an explanation about which of those continuous magnitudes are biologically important and how they support the "sense of magnitude." Thus, the target article fails to provide a sufficient explanation of what a generalized magnitude system entails.

Innateness of magnitude perception? Skill can be acquired and mastered at all ages

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Abstract: We agree with Leibovich et al.'s argument that the number sense theory should be re-evaluated. However, we argue that highly efficient skills (i.e., fluent and highly accurate, "automatic," performance) can be acquired and mastered at all ages. Hence, evidence for primacy or fluency in perceiving continuous magnitudes is insufficient for supporting strong conclusions about the innateness of this aptitude.

Leibovich et al. provide a critique of theories that posit an innate number sense. They propose that a "number sense" develops, via, for example, statistical learning, from the correlation between continuous magnitudes and numerosity. The authors argue that although numerosities are learned (through educational and cultural interactions), the perception of continuous magnitudes is innate. Thus, innate skills for the perception of continuous magnitudes set the stage for learning procedures for addressing discrete quantities.

Instead, we suggest that the evidence presented for (and against) the innateness of magnitude perceptions should be considered as addressing contrasts such as "primacy/no primacy" or even "automatic/not automatic" processing, in characterizing human numerical cognition at different phases of its development, rather than directly pertaining to innateness.

If innate means "not acquired" (e.g., Logan 1997), arguments for innateness and learning are mutually dependent. This is especially true when learning reaches a level wherein performance is "automatic" in the sense that it is fluent, is highly accurate, and exhibits a primacy in processing. Indeed, skilled automatized performance, especially when acquired implicitly, resembles innate processing. That is, both innate processing and automatized processing – perceptual, conceptual, or motor – are fast and may involve the involuntary direction of attention to stimuli and even, in some cases, a lack of conscious awareness (Karni 1996; Logan 1997). Specifically, both implicit learning and explicit learning may result in automatic processing of information that behaviorally is manifested in high levels of fluency. Fluency is reflected in the speed and accuracy of processin, as well as in a subjective experience of ease (Karni & Bertini 1997; Poldrack & Logan 1998). Thus, learning experiences can determine the saliency of a specific cue. Moreover, when a stimulus becomes salient, its salient (as opposed to the non-salient) features will automatically capture attention (Smith et al. 1996; Treisman & Gelade 1980), which in turn will further facilitate the learning process and enhance saliency. Therefore, saliency per se, even in early life

or in animal studies, is not sufficient for proving "innateness"; it may simply indicate the end point of a learning process.

In both innate and acquired skills, automatized processing occurs independently of top-down expectancies and thus leads to processing primacy. Consider, for example, the Stroop effect lending primacy to a complex and clearly acquired ability, reading (MacLeod 1991; Stroop 1935). The Stroop effect reflects the robust primacy of script processing (reading) even over simple color report in skilled readers. In the classic Stroop task, participants are instructed to name the color of the ink in which words denoting colors are printed. The Stroop effect refers to the fact that skilled readers cannot refrain from reading the words and, in fact, from accessing the meaning of the color words. Reading attains such primacy (automatic processing) that it interferes with the naming of (ink) colors. The primacy effect of reading can be found in other sensory domains. A recent study shows that tactile texture discrimination is interfered with by unintentional Braille reading of incongruent texture-denoting words in the blind (Jarjoura & Karni 2016).

There is evidence that automatic Stroop-like interferences develop with practice. For example, in a numerical Stroop-like effect that occurs when participants are presented with two digits that differ in physical size and numerical value and have to compare the digits using one of the dimensions, the interference between these two dimensions changes with practice (Tzelgov et al. 2000) and schooling (Girelli et al. 2000; Rubinsten et al. 2002). Thus, a numerical Stroop effect does not occur in physical comparisons at the beginning of first grade, but an adult-like pattern of the numerical Stroop effect was found in third grade and on (Girelli et al. 2000; Rubinsten et al. 2002). These data suggest that automatic activation of the numerical values of Arabic numerals develops, and attains primacy in processing, over the first years of schooling.

There is, moreover, ample evidence supporting the notion that very early implicit learning experiences – from visual and motor constrained environments (e.g., Held & Hein 1963) to cultural-social exposure (e.g., see review by Kuhl 2010) – can effectively generate and shape processing primacies. Consider in this light the processing primacy (bias) attained within even a few hours of exposure to a given visual environment, as attested by classical studies on dark-reared kittens (Held & Hein 1963). Very early life experience-dependent bias can even eliminate "innate" abilities as manifest in, for example, the finding that babies lose their ability to perceive multiple phonemic cues (sounds used in languages) that are irrelevant to their language experience before they attain 1 year of age (Eimas 1975; Eimas et al. 1971; Lasky et al. 1975; Werker & Lalonde 1988). On the other hand, by 10 months of age, language-specific differences can be discerned in the babbling of infants raised in different countries (de Boysson-Bardies 1993). The main question is not of innateness of perception or action, but rather how infants learn and form selective phonemic categories that make a difference in their language so early in life (Kuhl 2010).

To summarize, most of the evidence reviewed in Leibovich et al.'s article, including the interpretation of brain imaging studies, relates to the automaticity-primacy of processing continuous magnitudes (inferred from measures of fluency). This, we would argue, does not constitute sufficient evidence for determining the status of task performance as "a basic sense" or "innate" because implicit (and explicit) learning experiences in infancy (and later in life) can generate fluency, accuracy, and, importantly, primacy for processing specific cues. There is no a priori reason to suppose that the processing of continuous magnitudes or discrete quantities cannot reflect implicit learning experiences even from very early on in life. Indeed, it has been argued that in some situations constructs such as discrete and countable magnitudes may precede constructs of continuous magnitudes and may affect their development (Starr & Brannon 2015). We suggest that studies of biology–environment interactions, shaping our repertoire of automaticity, are warranted.

What is a number? The interplay between number and continuous magnitudes

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Abstract: Leibovich et al. argue that it is impossible to control for all continuous magnitudes in a numerical task. We contend that continuous magnitudes (i.e., perimeter, area, density) can be simultaneously controlled. Furthermore, we argue that shedding light on the interplay between number and continuous magnitudes – rather than considering them independently – will provide a much more fruitful approach to understanding mathematical abilities.

Leibovich et al. criticize the results of different studies employing non-symbolic numerical tasks, because the effect of continuous magnitudes would have not been adequately controlled. By definition, a non-symbolic number is the numerosity extrapolated from an array of elements (Feigenson et al. 2004). We agree that it is impossible to equate simultaneously both the overall area and the perimeter of two different arrays of elements, and that considering only the overall area is only a partial control. Nevertheless, Leibovich et al. did not consider that when the overall perimeter of two arrays of two-dimensional squares is equated, a negative correlation with the area occurs.

From a theoretical viewpoint, given a first array depicting a number $n_i \geq 1$ of squares of side l_{n_i}, and a second array depicting a number $n_j > n_i$ of squares of side l_{n_j}, it is impossible to simultaneously keep constant both areas $\left[\left(A_{n_i} = n_i l_{n_i}^2\right) = \left(A_{n_j} = n_j l_{n_j}^2\right)\right]$ and perimeters $\left[\left(P_{n_i} = n_i 4 l_{n_i}\right) = \left(P_{n_j} = n_j 4 l_{n_j}\right)\right]$ of the arrays. In fact, to do so, the following system has to be solved for l_{n_j}:

$$
\begin{cases}
l_{n_j} = \dfrac{n_i l_{n_i}}{n_j} & \text{when } P_{n_i} = P_{n_j} \\[2ex]
l_{n_j} = \dfrac{\sqrt{n_i} l_{n_i}}{\sqrt{n_j}} & \text{when } A_{n_i} = A_{n_j}
\end{cases}
$$

This system can be solved if and only if $n_i = n_j$, thus violating the hypothesis $n_j > n_i$.

We can now evaluate the relation between P_{n_i} and P_{n_j} when the overall area is kept constant (i.e., $A_{n_i} = A_{n_j}$).

$$
P_{n_j} = \frac{n_j}{\sqrt{n_j}} 4 \sqrt{n_i} l_{n_i}
$$

For any natural number $n_j > 0$, it holds true that $n_j > \sqrt{n_j}$. Moreover, if $n_i < n_j$, then $\frac{n_i}{\sqrt{n_i}} < \frac{n_j}{\sqrt{n_j}}$. Therefore, the size of P_{n_j} increases as a direct function of the increase of n_j whenever $A_{n_i} = A_{n_j}$.

We can finally evaluate the relation between A_{n_i} and A_{n_j}, whenever the overall perimeter is kept constant.

$$A_{n_j} = \frac{n_j}{n_j^2}\left(n_i l_{n_i}\right)^2$$

For any natural number $n_j > 0$, it holds true that $n_j < n_j^2$. Moreover, if $n_i < n_j$, then $\frac{n_i}{n_i^2} > \frac{n_j}{n_j^2}$. Therefore, it is evident that A_{n_j} will decrease as a function of the increase of n_j whenever $P_{n_i} = P_{n_j}$.

To sum up, number and perimeter are positively correlated when the area is kept constant, whereas number and area are inversely correlated when the perimeter is kept constant.

For example, we can draw two different scenarios:

1. A square of side 3 cm has an equivalent area (9 cm²) with respect to three squares of side 1,732 cm (3×3 cm²=9 cm²). However, the perimeter increases proportionally, shifting from 12 cm in the first case to 20,785 cm in the second case.

2. A square of side 3 cm has an equivalent perimeter (12 cm) with respect to three squares of side 1 cm (3×4 cm=12 cm). However, the area decreases proportionally, shifting from 9 cm² in the first case to 3 cm² in the second case.

This evidence prompts us to consider a hypothetical case. If you are in the first scenario and you choose the set of three squares rather than the single square, the choice can be ambiguously based on numerosity (3 vs. 1, respectively) or on perimeter (20,785 vs. 12 cm, respectively). However, if you are in the second scenario and you make the same choice, this is likely based on numerosity (3 vs. 1, respectively) rather than on area (3 vs. 9 cm², respectively). This is indeed a crucial test, enabling us to exclude the role played by continuous magnitudes (i.e., area and perimeter) in numerical tasks. However, we cannot exclude the influence of other crucial variables such as occupancy and density.

We contend that the simultaneous control for an entire set of continuous variables is not only possible, but also should be guaranteed, as recently demonstrated (Rugani et al. 2015). In this study, implying numerical discriminations in 3-day-old chicks, a simultaneous control for multiple variables was obtained: (1) perimeter (summation of the perimeters of all elements in each array was identical), (2) area (summation of the area of all elements was negatively correlated with numbers), (3) occupancy (the overall space occupied by each set of elements was, on average, the same for small and large numbers), and (4) density (the mean distance among the elements was not statistically different). Subjects' behavior revealed that only number was driving their actions, not perimeter, density, occupancy, or area.

Nonverbal subjects, thus, can master numerical tasks, purely on the basis of numerical information. But this is not the only evidence in the scientific literature. In some cases, human infants and young chicks were shown to be more sensitive to continuous magnitudes, rather than numbers (Clearfield & Mix 1999; 2001; Feigenson et al. 2002; Rugani et al. 2010).

Moreover, when both continuous magnitudes and numerical cues are available and consistent, human infants (Suanda et al. 2008) and nonhuman animals (Rugani et al. 2011; Stancher et al. 2015) can solve increasingly complex numerical tasks. This suggests that subjects can keep on hold and integrate multiple sources of information, so that redundancy can be exploited to increase mathematical abilities.

Why is scientific evidence on non-symbolic numerical cognition so puzzling? A possibility is that it reflects what a non-symbolic number is. By definition, a non-symbolic number is the numerosity extrapolated from an array of perceived stimuli. Leibovich et al. conceive non-symbolic numbers as abstract representations of discrete quantity (number). We believe, instead, that non-symbolic numbers are part of a more general system for representing quantity, discrete or continuous (see also Gallistel 2011).

To sum up, here we propose a new and more fruitful approach that can be helpful for controlling all of the continuous variables, to create tasks that can be solved solely on the basis of numerical (discrete) information. Controlling occupancy and density, and applying our algorithm to equate overall perimeter (obtaining, thus, a negative correlation between number of elements and overall area) would allow us to assess non-symbolic numerical competences, in human and nonhuman animals.

A future challenge in numerical cognition would be, then, to disentangle the relative role and weight of different cues (number and continuous magnitudes) and how they interact to influence estimation.

A "sense of magnitude" requires a new alternative for learning numerical symbols

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Abstract: Leibovich et al. proposed that the processing of numerosities is based primarily on a "sense of magnitude." The consequences of this proposal for how numerical symbols acquire their meaning are, however, neglected. We argue that symbols cannot be learned by associating them with a system that is not yet able to derive discrete numbers accurately because of immature cognitive control.

Leibovich et al. argue that the processing of continuous magnitudes is more automatic than the processing of (discrete) number. Such a claim is in line with some of our recent studies in which we repeatedly observed that decisions on discrete number are inevitably influenced by continuous magnitudes (e.g., Gebuis & Reynvoet 2012b; Smets et al. 2015). Despite describing slightly different mechanisms of how number can be inferred from continuous magnitudes – that is, either through "the mediation of inhibition" as proposed in the current review (Leibovich et al.) or through "sensory integration of continuous magnitudes" (Gebuis et al. 2016) – both reviews basically put forward the same conclusion. The take-home message of both is that the ability to extract discrete number originates from the ability to process continuous magnitudes, a finding that consequently seriously questions the existence of an approximate number system (ANS) as an evolutionary ancient system.

Besides the consequences of this conclusion for numerosity processing being addressed in the review of Leibovich et al., such a proposition also has serious implications for symbolic number (e.g., verbal number words, digits, etc.) processing. In particular, the *symbol grounding problem* (i.e., how symbols acquire their numerical meaning) needs to be revisited in light of these new arguments (Reynvoet & Sasanguie 2016). The presumed idea of the existence of the ANS has resulted in the proposal that when children are confronted with symbolic numbers for the first time, they acquire the numerical meaning of these symbols by mapping them onto this ANS (e.g., Dehaene 2001; Halberda et al. 2008). In turn, because these symbols are

crucial for learning formal mathematics (e.g., calculation), the assumption of symbol–ANS mapping has inspired researchers to hypothesize and observe that the performance on a numerosity discrimination task (which is assumed to index the ANS) is related to individual differences on a (symbolic) mathematics achievement test. In other words, following this assumption, the ANS would thus serve as the basis of symbolic numbers and also later symbolic arithmetic. Leibovich et al. (abstract), however, suggest that "continuous magnitudes are more automatic and basic than numerosities," and that we need to derive discrete number from these continuous magnitudes (see also Gebuis et al. 2016). If this is the case – which we believe it is – then the assumption that we learn symbolic numbers by mapping them on such a magnitude system seems very unlikely.

The present review suggests in particular that to derive discrete number from visual sets, we must apply cognitive control. It is well known that cognitive control develops through childhood (Anderson 2002). The continuing development of cognitive control functions like inhibition is also manifested in the performance of young children on a numerosity discrimination task. For example, when examining the performance on incongruent trials (i.e., continuous magnitudes correlate negatively with number) in 7-year-old children, Szücs et al. (2013) observed a performance at chance. Defever et al. (2013) demonstrated that the congruency effect (i.e., the performance difference between congruent and incongruent trials) in children ranging from 6 to 11 years old decreased with increasing age. These studies demonstrate that the ability to infer number from continuous magnitudes, just like cognitive control functions, develops during elementary school. In contrast, children's knowledge and understanding of numerical symbols (i.e., digits) is completed earlier. For example, Sasanguie et al. (2012) demonstrated that 5-year-old kindergartners are able to compare digits with an accuracy far above chance. By the time children finish second grade, not only are they almost perfect in symbolic number discrimination (Sasanguie et al. 2012), but also they master several symbolic calculation skills (e.g., multiplications), whereas they still encounter problems deriving number from visual sets (e.g., Szücs et al. 2013). In sum, when symbolic numbers are learned, the ability to infer number from the magnitude system has not reached its full potential yet – because of its dependence on the development of cognitive control functions. More specifically, children still struggle with inhibiting continuous magnitudes (Leibovich et al.) or integrating this sensory information (Gebuis et al. 2016), resulting in inaccurate judgements of number on stimuli where continuous information negatively correlates with number. As a consequence, it unlikely that such a magnitude system is the ground for learning symbolic numbers.

In our recent paper (Reynvoet & Sasanguie 2016), we evaluated a promising alternative for symbol grounding. In a first step, numerical symbols are mapped on the object tracking system, a system that allows us to keep track of up to four items (see also Carey 2009). In a second phase, knowledge about the counting list may be used to infer critical principles of the symbolic number system, such as the principle that numbers later in the counting list are larger (Davidson et al. 2012). As a result, gradually, symbolic numbers are primarily represented through (order) associations with other symbolic numbers in a separate semantic network of symbolic numbers (Sasanguie et al. 2017). Very recently, it was demonstrated through computational modeling that classic effects in numerical cognition, like the distance and the size effect, can be accounted for by such a network (Krajcsi et al. 2016).

In sum, the current review rightly identifies some serious challenges for the idea that we have an innate sense of numbers. An issue that has been overlooked, however, concerns the fact that this re-evaluation of the ANS into an "approximate magnitude system" (AMS) has important implications for symbolic number processing: It is impossible that symbols are learned by associating them with a magnitude system that, at the time of symbol acquisition, is not able to compute discrete number.

Numerical intuitions in infancy: Give credit where credit is due

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Abstract: Leibovich et al. overlook numerous human infant studies pointing to an early emerging number sense. These studies have carefully manipulated continuous magnitudes in the context of a numerical task revealing that infants *can* discriminate number when extent is controlled, that infants fail to track extent cues with precision, and that infants find changes in extent less salient than numerical changes.

In presenting their case for an *acquired* number sense dependent upon continuous magnitudes, Leibovich et al. overlook a substantial literature revealing that continuous magnitudes – across sets of multiple objects – are relatively difficult to track (e.g., Barth 2008). Importantly, the authors rely primarily upon data from adults to support their developmental model, a population shown to invoke advanced estimation strategies while concurrently shying away from more intuitive strategies (Siegler & Booth 2005). As such, data from adults are more likely to reflect learned notions and strategies with regard to number, not intuitions. Evidence from early human development is required to appropriately address questions of inherent abilities. Notably, however, the authors disregard numerous findings with preverbal infants contradicting their claims of the primacy of continuous magnitudes. In particular, these studies reveal that infants (1) *can* discriminate number when continuous extent is controlled, (2) are relatively *poor* discriminators of continuous extent, and (3) are more likely to attend to *changes in number* compared with changes in continuous extent. Similarly to Leibovich et al., we review evidence only pertaining to large, non-symbolic numerosities (>4).

First, although it is true that infant abilities to detect numerical changes can be influenced by continuous extent cues (e.g., Cantrell et al. 2015a), habituation studies reveal infants are capable of tracking number independent of these cues. The presence of purely numerical abilities early in development is consistent with a number sense account. Leibovich et al. discount infant habituation methods, arguing that inherent confounds between number and extent prevent studies from isolating numerical abilities. However, the authors fail to acknowledge a number of sophisticated designs that have taken advantage of the initial habituation phase to dissociate number and continuous extent (Xu & Spelke 2000). In these designs, infants are habituated to a series of exemplars of a single numerosity while extent cues are systematically varied; in subsequent test trials, the degree of change in extent across novel and familiar number test trials is equated. As such, these designs allow for one and only one dimension of discrimination: number. Under these circumstances – when extent is not a reliable cue for discrimination – infants have repeatedly succeeded in detecting changes in number (e.g., Lipton & Spelke 2003; 2004; Xu & Arriaga 2007; Xu & Spelke 2000). If early numerical abilities were entirely dependent upon continuous magnitudes, these robust findings would not be possible.

Second, infants track continuous extent with relatively poor precision. If it is the case that infants rely upon continuous magnitudes when comparing sets, then infants should be *at least* as good at detecting changes in continuous extent as they are at detecting changes in number. This is not the case. Studies that have examined infant abilities to track continuous extent across multiple items have consistently found that infants are remarkably limited in their ability to discriminate extent. In particular, 6- to 7-month-olds require as much as a 4-fold change in cumulative area

(Brannon et al. 2004; Cordes & Brannon 2008) or individual element size (Cordes & Brannon 2011) and a 3-fold change in contour length (Starr & Brannon 2015) in order to successfully detect a change in these continuous variables. In contrast, parallel studies isolating numerical abilities reveal that infants of this age reliably discriminate strikingly smaller changes in number (2-fold; Xu & Spelke 2000). That is, infants notice 2-fold changes in number (when extent is deconfounded) but require the size of individual items in an array to *quadruple* in size in order to detect a change in extent. Although infants are capable of tracking continuous extent, they are remarkably poor at doing so.

Third, infants find numerical changes to be more salient than extent changes. If infants relied upon continuous extent representations for tracking quantity in the world, this would entail not only a precise ability to track changes in extent, but also a greater *proclivity* to notice these changes over changes in number. Looking-time studies directly pitting changes in number against changes in continuous extent, however, demonstrate that infants more readily attend to numerical changes. Cordes and Brannon (2009) pitted number against contour length and cumulative surface area for large sets and found that in both cases 7-month-old infants dishabituated to changes in number but not to changes in continuous extent. Additionally, Libertus et al. (2014) found that even when a 1:3 ratio change in number was pitted against a 1:10 ratio change in cumulative area in a change detection paradigm, infants still did not show a preference for continuous extent, despite this dramatically larger change in the cumulative area. If numerical intuitions rested upon continuous magnitudes, infants would more readily notice changes in extent. They do not.

In sum, we argue that a careful reading of the infant literature provides strong support for an intuitive number sense. The finding that infants can track continuous extent in numerical tasks does not undermine evidence from other studies revealing that infants attend to number independent of extent. Yet preverbal infants' robust successes on numerical tasks do destabilize claims of an acquired number sense. Moreover, data revealing that infants are relatively poor discriminators of continuous extent and that infants find numerical changes more salient than extent changes weaken assertions that magnitude tracking underlies numerical processing, posing a significant challenge to the neo-Piagetian model posed by Leibovich et al. Whether, over the course of development into adulthood, representations of number and magnitude become more closely intertwined remains to be determined; however, we believe that the infant data firmly point to an early and intuitive number sense.

From continuous magnitudes to symbolic numbers: The centrality of ratio

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Abstract: Leibovich et al.'s theory neither accounts for the deep connections between whole numbers and other classes of number nor provides a potential mechanism for mapping continuous magnitudes to symbolic numbers. We argue that focusing on non-symbolic ratio processing abilities can furnish a more expansive account of numerical cognition that remedies these shortcomings.

This commentary was motivated by two shortcomings of the target article by Leibovich et al.: First, its sole focus on whole numbers leaves out entire classes of numbers, such as fractions, that are integral to cultivating robust numerical understanding among children and adults (Siegler et al. 2011). Second, it does not offer a mechanism whereby continuous magnitudes can be linked to specific whole numbers. Subsequently, we argue that focusing on non-symbolic ratio processing abilities might furnish a more expansive account of numerical cognition, providing perceptual access to both twhole number and fraction magnitudes. Moreover, a ratio-focused account can provide a potential mechanism for mapping analog representations of continuous magnitudes to symbolic numbers.

Recent research has begun to systematically detail the ability of humans and other animals to perceive non-symbolic ratios (e.g., Jacob et al. 2012; Matthews et al. 2016; McCrink & Wynn 2007). Instead of focusing on individual non-symbolic stimuli in isolation, this work focuses on perceiving ratio magnitudes that emerge from pairs of these stimuli considered in tandem (Fig. 1a). As the extent of non-symbolic ratio processing abilities becomes clearer, some have called for research that foregrounds ratio perception as a possible basis for numerical cognition more generally (e.g., Matthews et al. 2016). Indeed, in a recent book chapter, Leibovich et al. (2016a) posited that the development of non-symbolic ratio perception "might be at the background of all other numerical developmental processes" (p. 370). In recognition of this fact, we view this commentary as an extension of the authors' own logic to address key gaps in the theory as presented in the target article.

First, we argue that whole numbers are not the whole story. In presenting their integrated theory of numerical development, Siegler et al. (2011) lamented that the field's focus on whole numbers has deflected attention from commonalities shared by both whole numbers and fractions. This is a particularly interesting point given that recent research has highlighted multiple commonalities in the ways we process different classes of number. To name a few:

1. Whole numbers and fractions have both been associated with size congruity effects (Henik & Tzelgov 1982; Matthews & Lewis 2016).

2. Processing of both whole numbers and fractions recruits the intraparietal sulcus (IPS) (Jacob et al. 2012; Piazza 2010).

3. Whole numbers and fractions can both be represented as magnitudes on number lines (e.g., Siegler et al. 2011).

4. Processing fractions and whole numbers exhibits distance effects in both symbolic (DeWolf et al. 2014; Moyer & Landauer 1967) and non-symbolic (Halberda & Feigenson 2008; Jacob & Nieder 2009) forms.

This last fact results because numerical processing obeys Weber's law, and this has two very important implications. The first was perhaps stated best when Moyer and Landauer (1967) wrote that observed distance effects for symbolic numbers implied that it "is conceivable that [numerical] judgments are made in the same way as judgments of stimuli varying along physical continua" (p. 1520). The second is a corollary to the first and seems widely unappreciated: Weber's law is fundamentally parameterized in terms of ratios between stimulus magnitudes. Ironically, even the way we represent whole numbers is governed by the ratios among them. Together, these points raise considerable potential for integrating the psychophysics of perception with numerical processing via the conduit of ratio.

Furthermore, we argue that non-symbolic ratio lays the foundation for a pathway to understanding all real numbers. Leibovich et al.'s theory in the target article bears interesting parallels with Gallistel and Gelman's (2000) theory that the primitive machinery for representing number works with real number magnitudes. The missing link for both is a compelling mechanism for establishing a correspondence between continuous non-symbolic magnitudes and specific number values. Herein lies the power of non-symbolic ratios. By juxtaposing two quantities instead of
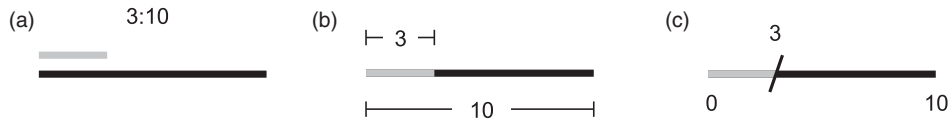
Figure 1. (Sidney et al.). Demonstration of the similarities between non-symbolic ratios made of line segments and number lines. From left to right, the panels represent (a) an example of a non-symbolic representation of the ratio 3:10 (or 10:3) based on stimuli from Vallentin and Nieder (2008), (b) the superimposition of the component stimuli of the ratio onto one line, and (c) how the addition of symbolic anchors yields the traditional number line estimation task. At a minimum, accurate number line estimation requires cross-format proportional reasoning, matching the symbolic 3/10 to a corresponding non-symbolic ratio.
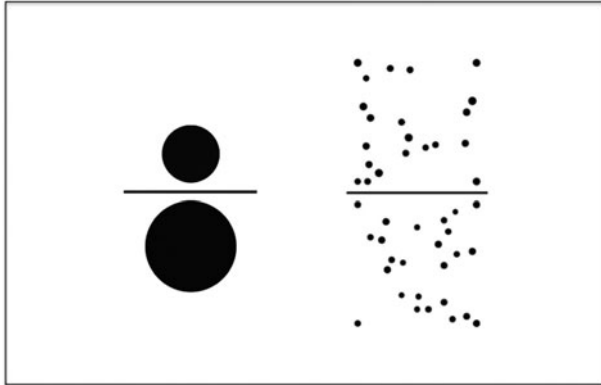

Figure 2. (Sidney et al.). Matthews and Chesney (2015) found that participants could accurately compare non-symbolic ratios across different formats in about 1,100 ms – even faster than they could compare pairs of symbolic fractions. This ability to compare ratios across formats implies that participants could perceptually extract abstract ratio magnitudes in an analog form.

one, ratios of non-symbolic stimuli can be used to indicate specific values. Although neither the gray nor the black line segments presented in Figure 1a correspond to a specific number, the ratio between the two corresponds only to 3/10 (or 10/3). Thus, non-symbolic ratio provides perceptual access to both fractions *and* whole numbers. In fact, because the components are continuous, these non-symbolic ratios can be used to represent *all real numbers*. In this way, non-symbolic ratios provide a flexible route for mapping non-numerical stimuli to specific real number values.

The potential of this conceptualization becomes clearest when we consider that competent number line estimation (i.e., linear estimates) can be seen as a task bridging symbolic and non-symbolic proportional reasoning (e.g., Barth & Paladino 2011; Matthews & Hubbard, in press). Indeed, Thompson and Opfer's (2010) use of progressive alignment with number lines to improve children's symbolic number knowledge can be interpreted as a case in which non-symbolic ratio perception is used to facilitate analogical mapping that endows unfamiliar symbolic numbers with semantic meaning. This technique leverages the fact that 15:100 is the same as 150:1000 in that both are the same proportion of the way across the number line, a fact that can help children understand the way the base-10 system scales up. Given that non-symbolic ratio perception is abstract enough even to support comparisons between ratios composed of different types of stimuli (e.g., Matthews & Chesney 2015) (Fig. 2), the possibilities for such analogical mapping abound. It may be that much of the psychophysical apparatus that operates in accord with Weber's law can be used to ground numerical intuitions. A focus on ratio processing stands to firmly situate numerical development within the generalized magnitude system proposed by the target article.

A comprehensive theory of numerical development should account for the deep connections between whole numbers and other classes of number, while accounting for relationships between symbolic and non-symbolic instantiations of numerical magnitudes. Leibovich et al.'s theory as presented in the target article neither accounts for numbers like fractions nor accounts for how continuous magnitudes can be mapped to specific numbers. However, adding a correction carving out a pivotal role for non-symbolic ratio perception might help provide the basis for a unified theory of numerical cognition.

Computational foundations of the visual number sense

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Abstract: We provide an emergentist perspective on the computational mechanism underlying numerosity perception, its development, and the role of inhibition, based on our deep neural network model. We argue that the influence of continuous visual properties does not challenge the notion of number sense, but reveals limit conditions for the computation that yields invariance in numerosity perception. Alternative accounts should be formalized in a computational model.

Numerosity perception is a key aspect of the number sense, and it is thought to be supported by a specialized mechanism, the approximate number system (ANS), which in primates has a specific neural substrate in the intraparietal sulcus (Nieder & Dehaene 2009). The finding that continuous visual properties influence numerosity judgments is used by Leibovich et al. as a main argument to claim that numerosity is processed holistically with continuous magnitudes. Their hypothesis that people do not extract numerosity independently from continuous magnitudes, as well as the related claim that perceived numerosity is simply the result of weighting a variety of continuous visual properties (Gebuis & Reynvoet 2012b), challenges a central tenet of the ANS theory and the notion of number sense more generally. However, this hypothesis is not grounded in any formal (mathematical or computational) model: In particular, it lacks any details about which continuous properties are necessary and sufficient to estimate numerosity, as well as how these continuous properties are extracted from the visual display in the first place.

Together with the apparent circularity in the statement that "number sense develops from understanding the correlation between numerosity and continuous magnitudes" (Leibovich et al., sect. 8, para. 6), this leads to a "non-numerical" account of numerosity perception that does not seem to have the explanatory value that one should expect from a cognitive theory.

The nature of the mechanisms underlying numerosity perception has been debated for decades (e.g., Allik & Tuulmets 1991, Burr & Ross 2008; Dehaene & Changeux 1993; Durgin 1995), and the fact that numerosity perception can be non-veridical has been known even longer (e.g., Frith & Frith 1972). However, recent computational modeling work based on unsupervised learning in "deep" neural networks (see Zorzi et al. [2013] and Testolin & Zorzi [2016] for a review of the approach) has provided a state-of-the-art and neurobiologically plausible account of how visual numerosity is extracted from real images of object sets. Stoianov and Zorzi (2012) showed that numerosity emerges as a high-order statistical property of images in deep networks that learn a hierarchical generative model of the sensory input. Learning in the network only involved "observing" images, and it aimed at efficient coding of those images, without providing any information about numerosity (i.e., there was no teaching signal). As a result of this unsupervised learning, number-sensitive neurons emerged in the deepest layer of the network, with tuning functions that mirrored those of biological neurons in the monkey parietal cortex (Roitman et al. 2007). In agreement with the ANS hypothesis, the numerosity signal encoded by the population of number-sensitive neurons in the model was found to be largely invariant to continuous visual properties, and it supported numerosity estimation with the same behavioral signature (i.e., Weber's law for numbers) and accuracy level (i.e., number acuity) of human adults. Preliminary analyses of learning trajectory in the model also revealed good match to developmental changes in number acuity in infancy and childhood (Stoianov & Zorzi 2013).

Detailed analysis of the emergent computations in the Stoianov and Zorzi (2012) model showed that numerosity is abstracted from lower-level visual primitives through a simple two-level hierarchical process that exploits cumulative surface area as a normalization signal (Fig. 1). Contrary to Leibovich et al.'s "holistic" hypothesis that the number sense develops on the basis of a "sense of magnitude," the essential primitive in the emergent computations is not a continuous property but high-frequency spatial filters (implemented by center-surround neurons) that discretize the visual input. Note that the key role of high-frequency spatial filtering has been independently highlighted by Dakin et al. (2011) in their psychophysical model. In summary, visual numerosity is a high-order summary statistic in the Stoianov and Zorzi (2012) model, but this is the result of hierarchical non-linear computations rather than a simple weighted combination of continuous visual properties. Accordingly, numerosity comparison turns out to be impossible when the raw image is the only input to the decision (even when trained using machine learning algorithms; see Stoianov & Zorzi [2012]). Nevertheless, the emerged hierarchical mechanism is relatively simple and this fits well with the long phylogenetic history of the visual number sense (from fish [Agrillo et al. 2012], to primates [Brannon & Terrace 1998]).

The Stoianov and Zorzi (2012) model also suggests that the normalization process embedded into the mechanism extracting abstract numerosity may be inefficient in particular circumstances, such as when a strong manipulation of continuous visual cues generates high uncertainty (low signal-to-noise ratio) for the numerosity judgment (e.g., Fig. 3B in Leibovich et al.), and this effect is exacerbated by pathological conditions that affect inhibitory processing. This crucial insight can be illustrated with the combined behavioral-computational investigation of Cappelletti et al. (2014), which showed that the decline of number acuity in elderhood (Halberda et al. 2012) was limited to stimuli in which numerosity is incongruent with cumulative
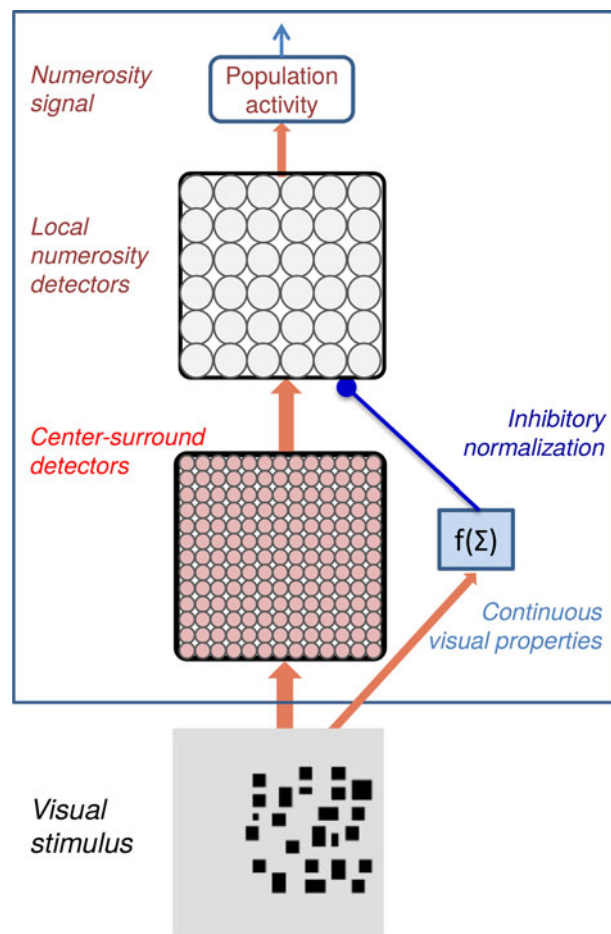

Figure 1. (Stoianov and Zorzi). Computational model of numerosity perception based on a hierarchical (i.e., deep) neural network architecture (Stoianov & Zorzi 2012). A layer of center-surround detectors with small receptive fields discretize the visual input and provide their signal to another layer of neurons with larger receptive fields. These number-sensitive neurons compute a local numerosity signal invariant of perceptual properties by means of inhibitory normalization. The population activity of number-sensitive neurons provides the final numerosity signal.

surface area. In turn, this effect was linked to the (in)efficiency of inhibitory processing, as indexed by performance in classic cognitive control tasks (e.g., Stroop paradigm). Simulations with the Stoianov and Zorzi (2012) model revealed that degraded synaptic inhibition, which specifically affected inhibitory normalization in the network (Fig. 1), induced impaired comparison performance for incongruent stimuli while preserving performance on congruent stimuli, thereby accounting for the data in elderly humans. Note that the notion of inhibitory normalization in the Stoianov and Zorzi (2012) model is by no means equivalent to the hypothesis that inhibition is required to suppress irrelevant continuous properties at the decision level (as in Leibovich et al.), although we do not a priori exclude this additional effect. Inefficient normalization might also be involved in the atypical performance of children with developmental dyscalculia (Bugden & Ansari 2016; Piazza et al. 2010).

In conclusion, state-of-the-art computational modeling reveals that numerosity perception is supported by an emergent neuro-computational mechanism (implementing the ANS) that cannot be reduced to a simple combination of continuous visual properties. In contrast to the proposal of Leibovich et al., our modeling work shows that (1) the emergence of number sense is the result

of a learning process, but it does not hinge upon a pre-existing "sense of magnitude" or the availability of numerical labels; (2) numerosity can be abstracted from continuous magnitudes; and (3) the influence of continuous visual properties on numerosity judgments simply taps the limits of the normalization process that is embedded into the ANS. Any alternative theoretical account, including that of Leibovich et al., should be implemented as a formal (computational) model and compared to that of Stoianov and Zorzi (2012) in terms of descriptive adequacy, as it is current (and best) practice in other cognitive domains.

NOTE
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Selecting the model that best fits the data

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Abstract: Leibovich et al. argue that that none of the experiments they review really establishes that human adults, infants, or nonhuman animals are sensitive to numerosity independent of a range of continuous quantities. We do not dispute their claim that the empirical record is inconclusive but argue that model-based data analysis does offer a way to make progress.

Leibovich et al. review a series of experiments supposedly demonstrating the existence of a "number sense" in human adults, infants, or nonhuman animals, and argue that none of them is really conclusive. Although we tend to agree with this general conclusion, we find that the authors's alternative theory is on equally uncertain ground. In the following, we first critically discuss some of the arguments the authors put forward, before concluding that data analysis based on precise, computational models does offer a more promising way to make progress.

The authors go into considerable detail in explaining why previous methods for detecting the number sense are flawed. Their critique on the stimuli centers around the inability to uncouple numerosity from various continuous variables, like surface area, density, object size, and convex hull size. Their critique on the measured variables, moreover, is that what is measured could often be something else than pure "number perception" itself, like, for example, the "summation of object area."

However, the authors' casual use of the term "continuous variables" deserves some scrutiny. In their description, they make no distinction between for example object size, convex hull size, cumulative surface area, outline, and density. Distinctions between these quantities are, although often subtle, very relevant. For example, surface density (How many pixels are black vs. white per surface area?) differs from object density (How many objects are there per surface area?), as their formulas show: To get surface density, one simply divides the total surface area of all objects (black) by the total area of white space. To arrive at the object density, one divides the number of objects by the total surface. Note that to do this, one needs to know both the total surface, or convex hull, and the number of objects. That means that to perform the calculation that gives you object density, numerosity

is required! Therefore, in studies in which object density is the "continuous variable" that participants could have used to estimate numerosity, they may very well have been using a derivative of numerosity.

An even bigger issue is the authors' implicit assumption that the processing of continuous variables contradicts a number sense. But even if the authors are right in claiming that continuous variables play a role in many of the cited experiments, that is not grounds to conclude a "number sense" does not exist. The existence of one mechanism does not exclude the existence of another.

Does that mean that no conclusions about the discrete or continuous, or innate or learned nature of number cognition are possible? No, by developing precise models of both, we can say much more about these issues than we currently can, as demonstrated in other domains of cognitive science that once seemed stuck in unproductive debates about nurture versus nature (e.g., Borensztajn et al. 2009) or continuous versus discrete variable accounts (e.g., Westermann 2000; Zuidema & de Boer 2014).

Model-based data analysis would allow us to evaluate whether the continuous factors can play a consistent explanatory role, or that only ad hoc combinations of these factors, different for each experiment, fit the data as well as the numerosity theory popular in the literature. This is particularly important in the case of animal experiments, which the target article claims are all inconclusive, but which play no role in the final part of the article where an alternative theory is laid out, despite the fact that the influence of language in that theory would lead to strong predictions about animal behavior. We might not be able to exclude all alternative hypotheses, but we can make statements about which model (when specified in sufficient detail) provides the better fit to the data.

Fortunately, some prior work exists (e.g., Stoianov & Zorzi 2012; van Woerkom 2016) providing ideas on how to build precise computational models for the different theoretical positions outlined in the target article. Stoianov and Zorzi's (2012) model is a type of neural network (a restricted Boltzman machine), trained on reproducing a binary input image. The model shows signs of encoding numerosity as well as cumulative surface area, and has been interpreted as evidence that a number sense might not be innate per se, but that it is an emergent property of (visual) perception. By looking at the mechanisms behind how such a model performs a numerosity detection task, we can form and test hypotheses about the nature of human and other animals' numerosity detection.

For example, Stoianov and Zorzi (2012) analyzed the connection weights in models that had been trained on numerosity estimation. In doing so, they found both "cumulative surface detectors" (responding to the continuous variable of surface area) and a different set of numerosity detectors (whose activity was correlated only with numerosity, not with surface area), showing that detectors for both quantities can emerge, work in tandem, and complement, rather than compete, with each other. These types of analyses are critical in forming our understanding of estimating numerosities and the use of the "number sense" concept.

But more modeling work in this area is required to explore the broader area of models similar to Stoianov and Zorzi's and embodying alternative positions. For example, in our replication of the model (Van van Woerkom 2016), we end up with networks that are equally good at the training task (reproducing images), but do so without any apparent numerosity detectors. That raises questions about the robustness of the "number sense as side effect" result and its explanatory power for empirical results in cognitive science (e.g., those relating to the influence of language). That said, this domain is a perfect example of a domain where we should report how well each model fits the data, rather than making binary choices between rejecting and accepting alternative theories. Modelers should get to work!

Controlling for continuous variables is not futile: What we can learn about number representation despite imperfect control

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http://faculty.missouri.edu/vanmarlek/DCL/vanmarle.html

Abstract: Leibovich et al. argue that because it is impossible to isolate numerosity in a stimulus set, attempts to show that number is processed independently of continuous magnitudes are necessarily in vain. I propose that through clever design and manipulation of confounding variables, we can gain deep insight into number representation and its relationship to the representation of other magnitudes.

Leibovich et al. breathe new life into an age-old question, which is whether discrete quantity representation (i.e., *number*) is on a par with, or a derivative of, the representation of continuous physical magnitudes (e.g., length, area, volume, loudness, pitch, color). The authors rightly point out that recent theorizing has favored a view (the "number sense" theory) in which number representation is innate and distinct from the representation of other magnitudes. However, their argument in favor of an alternative view (the "sense of magnitude" theory) falls short, having neither sufficient empirical support, nor logical soundness. These shortcomings stem from the authors' legitimate and veracious claim that because number out in the world naturally correlates with a host of non-numerical quantitative dimensions (e.g., a set of "four apples" has roughly twice the volume, twice the surface area, twice the weight, and twice the "redness," as a set of "two apples," not to mention other variables such as convex hull [perimeter of the set], density of the array, and amount of time it takes to visually scan the display), that it is "impossible to create two sets of items that differ in numerosity only" (sect. 3, para. 1). The implication here is that unless a researcher rules out *every possible alternative* in their study, they cannot be certain that responses were based on number per se, making it impossible to find pure evidence in support of the independence of number processing. The conclusion the authors draw from this – that lack of evidence for one theory ("number sense") means the alternative theory is true – however, is fallacious (ad ignorantiam).

Although Leibovich et al. are right that it is mathematically impossible to rule out all continuous variables simultaneously, I argue that this fact does not undermine efforts to control some dimensions while examining numerical judgments. Logically speaking, finding that anti-correlated features influence performance (as in Hurewitz et al. 2006, for example) does not necessarily mean that subjects' judgments are not also influenced by number. The interference may stem from the dimensions being represented together (as proposed in "sense of magnitude" theory), or they may stem from other sources (e.g., attention, response competition, etc.). Likewise, just because one cannot control all possible continuous dimensions, it does not follow that subjects therefore must be using the uncontrolled dimension(s). There are many examples in the literature and cited in this article (and at least two important studies left uncited – McCrink & Wynn [2004] and Izard et al. [2009]) in which the dimensions left to vary are clearly not driving performance because they predict different (and unobserved) patterns of performance than do judgments made on number. For example, through careful stimulus design Izard et al. (2009) were able to rule out infants' use of "intensive parameters" (e.g., item size, density) and "extensive parameters" (summed luminance, total surface area of the array). Specifically, infants' use of intensive parameters (which were equated across numerosities) predicted equal looking to all of the test images, while their use of extensive parameters predicted the same direction of preference across familiarization conditions. However, neither of these patterns was obtained – infants looked longer at the numerically matching display in both familiarization conditions. Thus, despite not having perfect control, the study design had strong discriminant validity, allowing the authors to conclude which of the three cues (numerosity, intensive quantities, and extensive quantities) were driving the observed looking time pattern. Indeed, the deliberate control and manipulation of competing variables allowed Izard et al. to provide what may be the strongest evidence to date for the "number sense" theory.

A related point is that contrary to the authors' claims, the "number sense" theory may actually be more parsimonious than the "magnitude sense" proposal. The authors seem to treat all continuous dimensions the same, as if they are equivalent (equally salient, equally informative, equally accessible, equally accurate, etc.). Given the specific study they are critiquing, the authors can claim subjects are using surface area or volume or density or perimeter, rather than number. Importantly though, because the specific dimensions being controlled (or not) vary from experiment to experiment, the explanation for what subjects are doing *instead* of number across this constellation of studies is necessarily complex. Therefore, it may be argued that appealing to number is really the more economical approach.

My final point is more general. I applaud the authors for tackling this critical issue. They are absolutely right that determining the nature of these systems is necessary for understanding the basis of human quantitative reasoning. The development of these systems must be more fully determined if we want to understand how early emerging abilities affect the early learning of symbolic mathematics. However, I do not agree that simply appealing to "continuous magnitudes" as if the various continuous dimensions are interchangeable is any better than ruling out only a subset of them and claiming that subjects were definitely using number! We know very little about whether various dimensions are treated similarly or differently and how the performance profile for each dimension changes over development. Perhaps a good course of action would be to step back and determine whether the "continuous magnitudes" actually hang together within the same system before deciding whether number does. Of course, this may not even be possible; continuous magnitudes tend to correlate not only with number, but also with each other, making it just as challenging to isolate surface area, for example, from all other dimensions, as it is to isolate number. Nonetheless, I would argue that the endeavor is still very much worth the effort. With clever manipulation of the variables available in a given stimulus set, researchers may be able to discover not only whether discrete and continuous quantities stem from the same or different cognitive sources, but also how they interact and influence each other, as they surely must do. Such insights will no doubt help us on our way to understanding how intuitive number and magnitude sense contributes to the crucial acquisition and development of symbolic mathematical skills and knowledge.

Authors' Response

Toward an integrative approach to numerical cognition

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Abstract: In response to the commentaries, we have refined our
suggested model and discussed ways in which the model could be
further expanded. In this context, we have elaborated on the role
of specific continuous magnitudes. We have also found it important
to devote a section to evidence considered the "smoking gun" of the
approximate number system theory, including cross-modal studies,
animal studies, and so forth. Lastly, we suggested some ways in
which the scientific community can promote more transparent
and collaborative research by using an open science approach,
sharing both raw data and stimuli. We thank the contributors
for their enlightening comments and look forward to future
developments in the field.

R1. Introduction

The main goal in writing the target article was to initiate a
broad discussion regarding the role of continuous magnitudes in numerical cognition. After reading the enlightening commentaries written by leading scientists in the field
of numerical cognition, as well as other fields, we feel that
this goal has been achieved. We thank all of the contributors
who took part in this discussion. Your input expanded our
knowledge, helped us to further sharpen and clarify our
theory, and provided further venues to explore.

In what follows, we summarize our suggested theoretical
model and, in the process, address concerns raised by some
contributors by clarifying some aspects of the model. Next,
we address some of the strongest evidence supporting the
existence of an innate sense of number. Then, we elaborate
on the role of continuous magnitudes and discuss their role
in processing non-numerical magnitudes in more detail.
Finally, we discuss ways in which our theoretical model
can be further developed and explored, in the context of
numerical cognition and beyond. We conclude with some
suggestions as to what we can all do as a scientific community to reduce some of the inconsistencies in the literature
of non-symbolic number processing.

R2. The suggested theoretical model

Our suggested model is based on an extensive review of the
literature. As many contributors rightfully mentioned,
more evidence is indeed needed to test the model
(**Content, Vande Velde, & Adriano [Content et al.]**);
Hyde & Mou; **Opfer & McCrink**; and **vanMarle**). We
put forward a model that, on the one hand, supplies testable predictions and, on the other hand, could be altered
according to the empirical data.

Our theoretical model describes the development of
understanding what a number is. What does it mean to
fully understand the concept of number? We suggest it
means to understand that number is the quantity of items
in a set that usually correlates with, but is independent

of, continuous magnitudes. Number is independent of continuous magnitudes because each quantity can appear in
infinite sizes; for example, the number 2 can refer to two
skin cells, two pens, or two continents. However, when
comparing 2 pens and 10 pens, 10 pens will usually take
more surface area than 2 pens. We suggest that understanding these correlations is an important building block
of numerical cognition.

According to the model, because of physical and mental
constrains (i.e., poor visual acuity and the inability to individuate), newborns cannot use number (at least not visually,
and it is problematic to use audition to test it, as we will
discuss subsequently). This is true until about the age of
5 months, when the ability to individuate develops. The
model is parsimonious in the sense that it does not
assume an innate mechanism for numerosity; nevertheless,
it does not reject the notion of such a mechanism. It is possible that there is an innate mechanism for detecting
number, but it cannot be used until other systems
mature, just as we are born with legs but cannot walk
until the skeleton and the motor system are developed
enough to support walking. Testing for the existence of
an innate number sense in newborns, however, is challenging because of the constraints mentioned in the target
article. Some researchers have suggested that cross-modal
studies or studies in different modalities, such as auditory
studies, can provide more insight into the innateness of
number (**Burr**; **Hyde & Mou**; **Libertus, Braham, &
Liu** [**Libertus et al.**; **Margolis**; **Olivola & Chater**; and
Savelkouls & Cordes). Importantly, presentation of auditory stimuli is usually serial. This means that the participant
is required to keep the representations of the stimuli active
in working memory. Because the working memory capacity
of newborns and infants is limited, conclusions of such
studies are limited. What can be learned from studies
with newborns and young infants (**de Hevia, Castaldi,
Streri, Eger, & Izard** [**de Hevia et al.**]; **Jordan,
Rinne, & Resnick** (**Jordan et al.**); Libertus et al., **Lourenco, Aulet, Ayzenberg, Cheung, & Holmes** [**Lourenco et al.**; and **Rugani, Castiello, Priftis, Spoto, &
Sartori [Rugani et al.]**, and Savelkouls & Cordes) is that
they are able to discriminate magnitudes, but not necessarily
numerosities (**Mix, Newcombe, & Levine [Mix et al.]**).

With the development of individuation ability, an infant
can notice discrete objects. However, "discreteness" does
not equal numerosity – the individuated items are not necessarily represented mentally as a quantity. We concur with
vanMarle that discreteness plays a role from the moment
it is noticed. It is probably taken into account together with
all other magnitudes because of the correlation between
numerosity and continuous magnitudes in the environment. However, is discreteness the most salient cue of
quantity? We think not. This is simply one of many other
cues. The saliency of discrete or continuous magnitude
depends on task demands. For example, although in
Piaget's classic number conservation task (Piaget 1952)
children fail to understand that continuous magnitudes
can change without changing the number of items, when
the question to the child is phrased differently, or if
M&Ms (round candies) are used instead of coins, children
do notice number (Calhoun 1971). The saliency of number
when incongruent with continuous magnitudes also
depends on the development of other cognitive abilities.
There are many components of inhibition, and different

methods of testing cognitive control and inhibition measure different components (Diamond 2013). Some cognitive abilities were found to be present from an early age and even before symbolic knowledge (**Opfer & McCrink** and **Sasanguie & Reynvoet**). We argue that these results do not exclude the possibility that the type of inhibition required to inhibit continuous magnitudes has not developed. We agree with the contributors' stating that more studies are required to clarify the exact role of cognitive control in the model, and we discuss some options raised by the commentators in section R5.3. Another factor that can affect the saliency of different magnitudes is the way that the stimuli are composed (i.e., a bottom-up component). There are many different ways to create dot stimuli, and the ratio between the magnitudes in two groups can affect the saliency of the different magnitudes. For example, if in one group the total surface area is 4 cm and the number of dots is 4, and in the other group the total surface area is 40 cm and the number of dots is 8, the total surface ratio (4/40=0.1) is physically more salient than the numerical ratio (4/8=0.5) and can affect performance more. We elaborate on this issue in section R4.2.

An important component contributing to the development of the number concept is language, specifically, the exposure of infants to number words. Giving different sets of items the same number word (three teddy bears, three candies, three dolls) focuses attention on number (Mix et al. 2016). As mentioned by **Opfer & McCrink**, some studies demonstrate that preverbal infants are able to learn rules based on number. Although this is true, being preverbal does not mean that one cannot recognize the meaning of words. Preverbal babies are able to understand the meaning of many words (Baldwin 1993), and animals such as dogs are able to understand the meaning of hundreds of words (Kaminski et al. 2004) even though they cannot utter them. Therefore, being preverbal does not contradict understanding the meaning of number words.

R3. Convincing evidence for the innateness of the number sense: Is the number faculty alive and kicking?

In this section, we address evidence supporting the notion that number sense is innate. The target article focused on comparison tasks in the visual modality, pointing out that it is impossible to eliminate the option that continuous magnitudes play a role in these tasks. Some of the contributors are still skeptical about this (**Nieder**, **Opfer & McCrink**, and **Rugani et al.**), putting forward cross-modal studies, animal studies, and studies on infants to support their claims. We extend this discussion to evidence suggested by these contributors.

R3.1. Cross-modal studies

Some contributors suggested that cross-modal studies with infants bypass the inherent confound between numerosity and continuous magnitudes (**Burr**, **Hyde & Mou**, **Libertus et al.**, and **Opfer & McCrink**). In such studies, infants are exposed to X and Y number of items in one modality (e.g., visual, tactile) and X number of items in a different modality (e.g., number of sounds). Some of

these studies have demonstrated that infants look at mismatched trials (i.e., X items and Y sounds) and matched trials (i.e., X items and X sounds) for different durations, supporting the notion that numerosity is innate. Looking closer into the literature of cross-modal studies, however, reveals a complicated yet fascinating body of evidence that should be further examined before cross-modal studies can be the "smoking gun" (Hyde & Mou) for the innateness of number.

There are several issues with cross-modal studies that restrict the assertion that numerosity is innate. These caveats were put forward more than a decade ago by Mix et al. (2002a). We focus on two main caveats: confound of number and continuous magnitudes and mixed results in cross-modal studies.

First, the use of cross-modal designs does not disentangle numerosity and continuous magnitudes. To illustrate, it takes more time to play three drumbeats than two. To keep the duration constant, one must change the rhythm – three drumbeats in a faster tempo than two. Hence, the match and mismatch that infants detect can be explained by detecting other attributes, such as rhythm, and not necessarily numbers. Indeed, in these studies a lot of effort was put into "controlling" continuous magnitudes (**Nieder**), but is it at all possible? The influence of continuous magnitudes could be reduced but not excluded, as we discussed in the target article (see also Mix et al. 2002a). Because in most of these studies the infants were 5 months old or older, it is possible that discreteness was noticed. To evaluate the role of continuous magnitudes in such studies, it is important to separate the data according to congruity between discrete and continuous magnitudes and investigate whether the same conclusions (significant difference between matched and mismatched looking time) hold in the different congruity conditions. The bottom line here is that controlling for continuous magnitudes by itself is not enough. It is important to properly demonstrate the effect of such control.

Second, the results of cross-modal studies are often contradictory. In some studies, infants are expected to look longer at matched trials (Izard et al. 2009; Jordan & Brannon 2006; Starkey et al. 1990). In other studies, infants are expected to look longer at the mismatched trials (Feigenson 2011; Féron et al. 2006; Kobayashi et al. 2004; 2005). Feigenson (2011) suggests that the interpretation could go either way, depending on the design of the study; in spontaneous preference (without any habituation), infants were found to look longer at matched trials, whereas in studies that included habituation and tested for violation of expectations, infants looked more toward mismatched trials. However, this does not explain how using the same design and the same analysis produce mixed, often contradictory, results. Moore et al. (1987) used a violation of expectation design previously used by Starkey et al. (1983). Although Starkey et al. found preference for matched trials, Moore et al. found the opposite: 7-month-old infants looked significantly longer at the mismatched trials. Using the same design as Moore et al. and the same analysis of Starkey et al. (1990), Mix et al. (1997) found again that infants preferred mismatched trials. The difficulties of replication cast some doubt on the interpretation of the results of cross-modal studies. As is the case with any study that involves newborns and pre-verbal infants, interpreting the results is highly

challenging. Pre-verbal babies and newborns are not small adults. Their vision, memory, attention span, and other cognitive skills are limited, which limits the design options of the study. The conclusions drawn from such studies should reflect the possible contribution of such cognitive differences between infants and adults to the results and take into account the difficulty of interpreting them.

In sum, there are many challenges in cross-modal studies: In addition to confounds of continuous magnitudes, it is difficult to interpret the results of studies that are nonreplicable under some conditions (Coubart et al. 2015; Mix et al. 1997) or whose replication predicts an opposite significant effect. What we can learn from the cross-modal literature is that the ability to discriminate between different magnitudes exists from birth (Izard et al. 2009). This ability might include relying on both number and non-numerical magnitudes and representing abstractly the difference between "more" and "less."

R3.2. Infant studies and the innateness of the number sense

Lourenco et al., **Savelkouls & Cordes**, and **de Hevia et al.** reviewed some studies conducted with infants as young as 4 months old that demonstrate early emergence of a number sense, suggesting that the number sense is innate. Putting aside confounds of continuous magnitudes that were already discussed in the target article, one cannot yet conclude that numerosity is innate. **Rubinsten & Karni** claim that it is impossible to separate between an ability being innate or an ability being learned to the point of automatic and fluent performance. Therefore, it is impossible to say that either number discrimination or continuous magnitude discrimination is innate; by the age of 4 months there are many interactions of an infant with the environment, and a substantial learning process could have taken place. Accordingly, what we measure at the age of 4 or 7 months could be the end result of such a learning process. Accordingly, Rubinsten & Karni rightfully suggest shifting the focus of the studies from talking about "innateness" to studying how the biology-environment interaction shapes number/magnitude representation.

R3.3. Animal studies

Beran & Parrish and **Agrillo & Bisazza** suggested that the question of innateness could be answered by studying nonhuman animals, and especially animals that have no experience with magnitudes at all. Imagine having an animal model that has never been exposed to any magnitude – not in audition, vision, touch, or any other modality. How would such an animal perform in a comparative judgment task? The initial preference of such an animal could tell us which magnitude is truly innate. However, is that at all possible? Can one eliminate an animal's access to all possible magnitude cues in all possible modalities? It is difficult to think of such a scenario because, for example, even eating for a longer time means that you will probably eat more and feel fuller, and the same goes for drinking. As long as it is impossible to divest an animal of all possible magnitudes, then the best we can do is to get as much information as we can from the process in which animals learn to use magnitudes.

Agrillo & Bisazza discussed a study in which 1-day-old fish were able to choose the larger group of fellow fish and

considered it as evidence of an innate number sense. The claim is that because the fish were viewed serially, there was no confound with continuous magnitudes. However, it could be argued that the fish responded to the amount of time it took them to see all of the fish, or the rhythm in which the fish were viewed, not necessarily their quantity. Therefore, what can be concluded is that some magnitude discrimination is innate. However, we cannot determine which magnitude is innate.

R3.4. What does it mean to "control" for continuous magnitudes?

Rugani et al. suggested that continuous magnitudes could be controlled. They considered two continuous magnitudes – perimeter and area – and demonstrated how these two magnitudes could be controlled through an experiment. Rugani et al. described the relations between numerosity, area, and perimeter in an elegant and simple way. Accordingly, number and perimeter were positively correlated when the area was kept constant, whereas number and area were inversely correlated when the perimeter was kept constant.

However, dissociating perimeter and area from numerosity ignores the possibility that participants rely on other magnitudes or even switch between them. Salti et al. (2017) detailed an elaborate taxonomy showing a more complex relationship between continuous magnitudes and numerosity. According to this taxonomy, the inter-correlations between the different continuous magnitudes make the dissociation between numerosity and all continuous magnitudes far from trivial.

Importantly, **Rugani et al.** acknowledge the importance of continuous magnitudes in numerical perception, stating that human infants and nonhuman animals can solve complex numerical tasks "when both continuous magnitudes and numerical cues are available and consistent" (para. 7), or in other words, when they are highly correlated. We, of course, embrace this notion as it underlies the target article. Moreover, we put forward the notion that to develop an understanding of numerical perception one has to relate to continuous magnitudes as an attribute and a feature and not as a confounding element.

Importantly, the use of the word "control" is misleading and meaningless. What does it mean to control continuous magnitudes? Does it mean to abolish the influence of continuous magnitudes? This option is impossible. Hence, "controlling" continuous magnitudes means different things in different studies. For **Rugani et al.**, for example, it means dissociating different continuous magnitudes in different trials. For others, it means equating different continuous magnitudes in different trials. The ambiguity of the word "control" makes it difficult to interpret the results of studies involving number and continuous magnitudes and to understand the limitations of such studies. Therefore, it is important to provide details of the measures taken in order to deal with the confound of number and continuous magnitudes (e.g., equating, dissociating, etc.).

R4. Are all magnitudes created equal?

Most of the literature in the field of numerical cognition pits numerosity against all continuous magnitudes, as if

they were one. Because continuous magnitudes are intercorrelated with one another, many studies chose to control a subset of them. However, are all continuous magnitudes equally important for number perception? Do we rely on one continuous magnitude more than we do others? These questions have two important implications. First, if continuous magnitudes do not all contribute equally to number perception, then choosing which continuous magnitude to manipulate can have a major impact on the results. Moreover, as suggested by several contributors (**Durgin**; **Gebuis, Cohen Kadosh, & Gevers** [**Gebuis et al.**; **Park, DeWind, & Brannon** [**Park et al.**]; and **van-Marle**), understanding the unique contributions of the different continuous magnitudes can help us go beyond quantifying their impact to characterizing their influence on number perception.

R4.1. Prominent continuous magnitudes in numerosity comparison tasks

Continuous magnitudes differ in the type of information they convey; for example, intrinsic magnitudes (i.e., diameter, circumference, and area) are indicators of item size. Intrinsic magnitudes can be calculated for a single item or a set of items. For example, one can calculate the area of a dot or the total area of three dots. In contrast, extrinsic magnitudes (i.e., density and convex hull – the smallest polygon containing all items) provide information on the size and spatial location of the items (Salti et al. 2017). Accordingly, extrinsic magnitudes that contain information about both individual items' size and the spatial relationship between the items have been suggested to have greater influence on numerical judgments.

One example for an extrinsic magnitude that has been demonstrated to directly affect number perception is density. More specifically, Durgin (1995) adapted participants to a large number of dots on one side of the visual field and to a small number of dots on the other side of the visual field. In the test stage, a patch of dots was presented to either the visual field adapted to high numerosities or that adapted to low numerosities. Participants were then asked to estimate the number of dots. The results revealed that estimates were affected by the type of adaptation: Adaptation to high numerosities yielded higher estimates of numerosities. This was evident particularly for high numerosities (more than 40). These findings led the authors to conclude that density affected the perception of numerosity.

Convex hull was also recently suggested as one of the most influential continuous magnitudes in numerosity comparison tasks. Gilmore et al. (2016) showed that the ratio of convex hull areas in a dot comparison task consistently influenced responses for all ages (5–20 years old) and at all stimuli display times (16, 300, and 2,400 ms), whereas the influence of total surface area was stronger in childhood but diminished with age. Accordingly, the authors highlighted the importance of controlling convex hull in numerosity comparison tasks. The mechanism by which convex hull might influence number perception is still unclear. Interestingly, convex hull has played a major role in the subitizing literature. One of the prominent theories about subitizing involves pattern recognition. Involvement of pattern recognition suggests that non-symbolic numerosities that are arranged canonically form a pattern that is

automatically translated to a numerosity (Mandler & Shebo 1982). For example, three dots in the form of a triangle are automatically perceived as three. Katzin et al. (2016) recently suggested that the shape of the convex hull could account for the different ranges of enumeration (subitizing, counting, and estimation).

R4.2. Is the relationship between continuous magnitudes and number static?

Even studies that acknowledged the tight relationship between numerosity and continuous magnitudes, and tried to account for it, assumed that this relationship is constant and not dynamic. However, we have already shown that this relationship could change because of context (Leibovich et al. 2015) or because of saliency (Salti et al. 2017)

Although Gilmore et al. (2016) found that convex hull was the most influential continuous magnitude, some studies have shown other magnitudes to be more influential. For example, in a numerosity comparison task, Leibovich and Henik (2014) used the ratio between five continuous magnitudes and numerosity magnitudes as predictors of response time in two different settings. In the first setting, the groups of dots appeared to the left and right of the center of the screen, at the same latitude. Under this condition, stepwise regression revealed that after numerosity ratio, total circumference was the most influential magnitude. In the second setting, however, the same dot stimuli and procedure were used, but the groups of dots appeared at different latitudes, so one appeared "higher" on the screen than the other did. Under this condition, density was the most influential among the continuous magnitudes. Importantly, unlike Gilmore et al.'s design, the design of Leibovich and Henik contained different levels of congruity that were not taken into account. This demonstrates that the influence of different continuous magnitudes may change because of difference in stimuli, setting, or context.

Task context can also influence the dynamics between number and continuous magnitudes. Leibovich et al. (2015) asked participants to compare either the number of dots (in the subitizing range, i.e., the numerical task) or their area (i.e., the continuous task). Half of the participants started with the numerical task and half with the continuous task. The order of the tasks affected performance: Participants who started with the continuous task did not show any congruity effect; namely, their comparative judgments of area were not affected by the number of dots. In contrast, comparative judgments of area, by participants who started with the number task, were affected by the irrelevant number of dots, as demonstrated by the significant congruity effect in the area task.

Another factor that can influence the relationship between continuous magnitudes and numerosity is the way in which the stimuli are constructed. This has recently been demonstrated by Salti et al. (2017). In this work, the authors used three different sets of stimuli. In one set, the average diameter ratio was equal to the numerical ratio. In the second set, the total surface area ratio was equal to the numerical ratio. In the third set, the convex hull area ratio was equal to the numerical ratio. In all sets, all five continuous magnitudes (average diameter, total circumference, total surface area, density, and convex hull) were either congruent with numerosity (in half of the trials) or

incongruent with numerosity (in the other half). Participants were divided into three groups. Each group saw only one set of stimuli and performed either a numerical task or a continuous task (as described in Leibovich et al. 2015). The results revealed that the way in which stimuli were constructed (e.g., equating average diameter ratio and numerical ratio) affected performance in both tasks. For example, in the set where average diameter ratio was equal to the numerical ratio, the congruity effect in the continuous task was larger than the congruity effect in the numerical task. In the set where convex hull ratio was equal to the numerical ratio, there was a very small congruity effect in both tasks. Importantly, the tasks included numerosities between 2 and 4 (i.e., in the subitizing range); hence, more studies using different ranges of number are needed to generalize this result to larger numerosities.

The study of Salti et al. (2017) demonstrates how different ratios between magnitudes may affect performance. More specifically, if, for example, the ratio between two numerosities is close to 0 (i.e., a very large difference) and the ratio between total surface areas is closer to 1 (i.e., a very small difference), then it is more likely that numerosity would be a more salient cue than total surface area and would be used by the participant to compare magnitudes. This could be the case that **Burr** describes: Burr cites the work of Cicchini et al. (2016), who presented participants with three dot patches and asked them to pick the odd patch. The authors reported that regardless of task instructions (to choose by a specific magnitude or freely), participants tended to rely on numerosity when choosing the odd patch. However, a closer examination of Cicchini et al.'s stimuli reveals that the numerical ratio was smaller (closer to 0) than the ratios of the continuous magnitudes, suggesting that numerosity was more salient. Because the most salient cue was numerosity, it cannot be generalized to conclude that number is always more salient than continuous magnitudes.

Another example of the dynamic relationship between number and continuous magnitudes is the task itself. Although in a comparison task we can see that more area, larger size, and so forth are indicators of larger numerosity, the opposite occurs in number estimation tasks (**Gebuis et al.**). When participants are asked to estimate the number of presented items, continuous magnitudes have the opposite influence: The quantity of smaller items is overestimated, whereas the quantity of larger items is underestimated. Although this finding has been reported many times (Cleland & Bull 2015; Gebuis & van der Smagt 2011; Ginsburg & Nicholls 1988), there is still no explanation for this phenomenon. One possible explanation is that we compare the relative area (**Sidney, Thompson, Matthews, & Hubbard** [**Sidney et al.**]) of the items to the total area (of the screen, for example). From our experience, we know that we can fit more small objects into the same area compared with larger objects, explaining why the quantity of large items is underestimated and that of small items is overestimated. This, however, is just a suggestion that should be empirically confirmed.

To conclude, the influence of continuous magnitudes is dynamic and depends on task, saliency, the stimuli themselves, and so forth. This complexity highlights the importance of studying these factors and understanding their role in non-symbolic number and size perception.

Progressing in this direction of research can deepen our knowledge of numerical cognition.

R5. Expanding the model

The suggested model puts forward new predictions that could be tested empirically. In this part, we discuss these predictions and ways to test them. In addition, we discuss new lines of research aimed at expanding the scope of the model.

R5.1. What can we learn from nonhuman animal studies?

As mentioned by **Beran & Parrish**, nonhuman animals have demonstrated a wide variety of magnitude-related behaviors. For example, parrots and chimpanzees were able to learn to associate a quantity with a specific label. This often required lengthy training (years), but nevertheless, it was possible. Chimpanzees that were trained to choose the larger number of items were able to do so using a variety of different continuous magnitudes. We concur with Beran & Parrish that there is a lot to be learned from such training studies in animals. The actual training process can be insightful: Would the learning curve be steeper if at first the stimuli were composed so that the correlation between number and continuous magnitudes would be high? Which incongruent continuous magnitude would affect the rate of learning the most? These are only some of the questions we could test. In the case of the chimpanzee study (Cantlon et al. 2009b), it would be interesting to analyze the congruity effect throughout the training because it is possible that during training the chimpanzees learned to inhibit the irrelevant continuous magnitudes. It has been demonstrated that nonhuman animals have some cognitive control abilities (Deaner et al. 2007), and it would be interesting to study the interaction of cognitive control abilities and magnitude processing abilities in animals.

R5.2. What can we learn from computational models?

Another line of studying the possible role of continuous magnitudes in non-symbolic number processing is by using computational models. These are mathematical models that do not require human or nonhuman participants. Instead, a computational model aims to "imitate" brain processes of a computation (like when comparing two non-symbolic numerosities). The model is validated if the computational results are similar to behavioral results.

The advantage of computational models is that they are independent of strategy and prior knowledge. However, in the context of non-symbolic number processing, the problem of the correlation between number and continuous magnitudes still exists. Accordingly, computational models have produced mixed results regarding the independence of number and continuous magnitudes. **Stoianov & Zorzi** found, in their computational model, both number and area detectors operating in cooperation but independently of one another. In contrast, **van Woerkom & Zuidema** attempted to replicate Stoianov & Zorzi's model but failed to find number detectors, casting doubt on the robustness of the number sense.

Because of the great potential of computational models, it is important to keep using this tool to study the approximate magnitude system (AMS) and our suggested model under different scenarios. For example, **Stoianov & Zorzi** suggested in their model that non-symbolic images undergo a normalization process before being enumerated by a numerosity detector. This normalization process allows the system to ignore the different continuous magnitudes of the to-be-counted items. This normalization process could be equivalent to the suggested role of cognitive control in our model. Namely, it could be that instead of being completely inhibited, continuous magnitudes are normalized. This possibility should be further examined.

R5.3. The role of cognitive control in numerical cognition

The role of cognitive control in our suggested model should be further tested. As suggested by **Merkley, Scerif, & Ansari** (**Merkley et al.**), cognitive control does not work in isolation. Instead, it works in conjunction with domain-specific knowledge. Merkley et al. raise the intriguing possibility that both top-down and bottom-up attention processes can divert attention toward the discrete aspects of non-symbolic stimuli (i.e., numerosity). For example, knowledge of number words may direct attention in a top-down manner toward the numerosity of a set; physical features of the stimuli (e.g., the range of numerosity) can divert attention in a bottom-up manner toward numerosity (like in small quantities in the subitizing range) or to continuous magnitudes (like in large quantities). In other words, top-down and bottom-up control processes may play a role in diverting attention toward different features of the stimuli.

R5.4. Levels of representation

An important question raised by **Odic** concerns the different levels of representation of number and continuous magnitudes: Do they share a common abstract representation? Or, do they have separate representations that share similar encoding or decision-making components? A recent study by Sokolowski et al. (in press) shed some light on this question. In a quantitative meta-analysis of more than 90 functional magnetic resonance imaging studies, Sokolowski et al. demonstrated that the representation of symbolic and non-symbolic numbers in the brain was distinct. However, the representation of continuous magnitudes (brightness, line length, area, etc.) was not distinguishable from that of non-symbolic numbers. These results suggest that non-symbolic numbers and continuous magnitudes share a common representation. Of course, one should always keep in mind that even if both non-symbolic numbers and continuous magnitudes activate the same brain regions, the pattern of activation might differ. Therefore, it is important to directly study activation patterns of continuous magnitudes and non-symbolic numbers in the regions found by Sokolowski et al.

R5.5. Acquisition of the symbolic number system

The current model is limited to processing of non-symbolic numerosities. The literature discusses two ways in which symbolic numbers are acquired, that is, the symbol grounding problem. The first is that symbolic numbers are acquired by mapping them into an existing approximate number system (ANS). The second possibility is that symbolic and non-symbolic numbers are learned independently of each other and influence each other reciprocally (Leibovich & Ansari 2016). **Sasanguie & Reynvoet** suggest that if number is not processed automatically, the former theory seems unlikely. They suggest that the initial stage of acquiring symbolic numbers is the mapping of small symbolic numbers (up to four) to the object tracking system, and that large numbers are not mapped onto the ANS. We agree and highlight that a prerequisite for this initial stage is an understanding that numerosity is independent from continuous magnitudes, for example, an understanding that the quantities of two ants and two firetrucks are equal, despite their vast differences in size. Combining our model with **Merkley et al.**'s suggestion, we argue that acquiring the first number words enhances attention toward numerosity, thereby allowing a child to map the first symbolic numbers. Importantly, learning the independency of numerosity does not mean that continuous magnitudes will not bias performance as we know they bias even adults (Leibovich et al. 2015).

R5.6. Acquisition of fractions and other types of numbers

Sidney et al. mentioned that our model can also account for acquisition of fractions and proportions, not only whole numbers. They suggest that infants learn about ratios and proportions from a single stimulus also. For example, in a pizza box with eight slices, when two are missing, you can assess how much pizza is left by comparing the area covered by pizza and the area that is not covered. The assessment may not be exact. Importantly, unlike Sidney et al., we argue that this assessment does not involve knowing the exact number of pizza slices (i.e., numerosity). We do agree with Sidney et al. that understanding ratios and continuous magnitudes are interconnected, and that because continuous magnitudes are processed relatively, they can be used to represent all real numbers, not only whole numbers (see also Leibovich et al. 2016).

R6. Math abilities and education

One of the most important questions regarding the practical implications of our model is educational: Can performance in non-symbolic number/area comparison tasks predict math abilities? The literature has mixed evidence regarding this issue. **Inglis, Batchelor, Gilmore, & Watson (Inglis et al.)** performed a *p*-curve analysis to evaluate the distribution of *p* values in studies examining the correlation between ANS and math abilities. Their results demonstrated a right-skewed distribution of *p* values; namely, four of nine statistically significant results had a *p* value greater than .025. The authors suggested that these results cast doubt on the relationship between ANS and math abilities.

We agree with the contributors that investigating whether there is a causal relationship between ANS and formal math abilities is important. We believe that the discrepancies regarding the existence of the correlation between ANS and math abilities in the literature stem

from several reasons. The first reason is the variance of stimuli that are being used. It has already been demonstrated that performing the same task with different stimuli produces different results (Clayton et al. 2015; Salti et al. 2017). For example, some studies (e.g., Bugden & Ansari 2016) demonstrated that only performance in incongruent conditions was correlated with math abilities. Accordingly, an asymmetry in the number of congruent and incongruent stimuli might affect the correlation between ANS and math ability. Second, different types of stimuli might encourage a strategy of relying on number or on continuous magnitudes, depending on their saliency (e.g., Cantrell et al. 2015). It is possible that ANS predicts math abilities differently at different ages; ANS is a good predictor early on and is not so good later. Namely, the correlation between ANS and math abilities is attenuated by age. What might be responsible for such a pattern of correlation? It is conceivable that during the first steps of formal education, children still rely on more informal strategies, like the correlation between number and continuous magnitudes, and therefore the correlation is stronger. With more formal math training, children rely less on a "number sense" or a "magnitude sense" and use more advanced strategies. Importantly, we suggest that informal strategies continue to be useful outside of the classroom in everyday situations (like choosing the fastest line in the grocery store).

To gain more knowledge regarding the relationship between the AMS and math abilities, a few steps should be taken. First, it is important to use the same set of stimuli and the same experimental setting (like similar presentation times, etc.). However, even if this is not possible, the minimum requirement should be to report in detail how the stimuli were created and what their physical properties were (congruity conditions, ratio between number and non-numerical magnitudes, etc.). Second, it is important to include age as a factor, to understand possible age-related changes. A good example for age-related changes is the study of Gilmore et al. (2016) that revealed the specific influence of convex hull and total surface area in children and adults.

R7. Beyond numerical cognition

Some contributors have suggested ways in which other fields can benefit from the AMS theory. **Gronau** tries to place numerical cognition on the continuum of domain-specific versus domain-general organization of size in the brain. The number sense theory suggests the number has a designated module and hence is domain specific. Combining number with continuous magnitudes is a domain-general view. Indeed, the notion that size can be an overarching principle of brain organization receives support from the finding that objects are organized in the ventral temporal cortex according to their size (Konkle & Oliva 2012). As Gronau suggests, the commonalities between numerical perception, language, and so forth still require research.

Olivola & Chater emphasized the implications for the field of decision making. They highlighted the connection between magnitude evaluation and assigning values in the decision-making process. For example, when traveling, one must decide whether to purchase luggage insurance.

To make this decision, one evaluates the probability of the luggage being stolen or lost against the cost of the insurance. This process involves estimation and comparison of probabilities and costs (i.e., magnitudes). Olivola & Chater further argue that the variance in the decision-making process indicates that a stable number sense module does not exist.

R8. Moving forward: A problem shared is a problem halved

So far, we have clarified some ambiguous issues in our suggested model. We have reviewed some evidence that is considered the "smoking gun" for the ANS theory and argued against the interpretations. We hope that these sections help the readers better understand our model and position. We then expanded further on the model, based on the valuable suggestions raised by the contributors. In addition, we discussed the implications of our model on education and other fields of psychology. It seems that there is a lot more work to be done to expand the model and confirm or refute it. Moving forward, we would like to suggest some ways in which numerical cognition research could be promoted.

1. Sharing all of the available information about non-symbolic stimuli could be of great value to all parties involved. For example, even if the researchers chose to define congruity only by total surface area, the information about congruity and the ratio between the other continuous magnitudes should be accessible, in an appendix or online supplementary material if not mentioned in the paper.

2. Agreeing on a standard way of reporting the properties of a set of stimuli being used will help with comparing across different studies.

3. Creating a database with stimuli from different experiments for all to access.

4. Sharing raw data in depositories like the Open Science Framework (https://osf.io) can enhance collaboration and transparency.

5. Publishing nonsignificant results can help negate publication bias.

6. Preregistration of methods and expected results can ensure that a design and analysis are theory driven. Namely, preregistration requires a researcher to declare, before starting the experiment, the method, the number of participants, which analyses will be used, and what the expected results and theoretical implications are. In this way, the interpretation is less likely to be result driven, and the likelihood of *p*-hacking and "fishing" will be reduced.

To conclude, we discussed our theory suggesting that continuous magnitudes are more basic and automatic representations than numbers, and that understanding the correlation between number and continuous magnitudes will allow us to eventually understand the concept of number – as a quality of the set that is independent of, but highly correlated with, continuous magnitudes. The contributors helped us refine this notion and suggested ways in which the model could be further improved and expanded. We agree that there is a lot more work to be done to confirm or refute the model. We have suggested some ways in which research on numerical cognition can be promoted. We are excited to continue working on improving this

model and look forward to seeing what future studies will bring.

NOTES

1. Tali Leibovich and Naama Katzin contributed equally to this work.

2. Maayan Harel was not available to contribute to the Response article and did not participate in writing it. Moti Salti was not involved in writing the target article, but contributed to the Response article and participated in writing it.

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[The letters "a" and "r" before author's initials stand for target article and response references, respectively]

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